On Generalized Gauge-Fixing
in the Field-Antifield Formalism

I.A. Batalin[1]

I.E. Tamm Theory Division

P.N. Lebedev Physics Institute

Russian Academy of Sciences

53 Leninisky Prospect

Moscow 119991

Russia

K. Bering[2]

Institute for Theoretical Physics & Astrophysics

Masaryk University

Kotlářská 2

CZ-611 37 Brno

Czech Republic

and

P.H. Damgaard[3]

The Niels Bohr Institute

Blegdamsvej 17

DK-2100 Copenhagen

Denmark

February 1, 2008

Abstract

We consider the problem of covariant gauge-fixing in the most general setting of the field-antifield formalism, where the action W and the gauge-fixing part X enter symmetrically and both satisfy the Quantum Master Equation. Analogous to the gauge-generating algebra of the action W, we analyze the possibility of having a reducible gauge-fixing algebra of X. We treat a reducible gauge-fixing algebra of the so-called first-stage in full detail and generalize to arbitrary stages. The associated "square root" measure contributions are worked out from first principles, with or without the presence of antisymplectic second-class constraints. Finally, we consider an W-X alternating multi-level generalization.

---

[1]E-mail: batalin@lpi.ru   [2]E-mail: bering@physics.muni.cz   [3]E-mail: phdamg@nbi.dk

1   Introduction

The field-antifield quantization formalism [1, 2, 3] has been given a substantial reformulation, which shows how it fits into a much more general scheme [4, 5, 6, 7]. The essential ingredient is a Grassmann-odd and nilpotent differential operator Δ that is symmetric,

$$\Delta \; = \; \Delta^T \; , \tag{1.1}$$

with respect to a transposition defined by

$$\int d\mu \; F(\Delta G) \; = \; (-1)^{\epsilon_F} \int d\mu \; (\Delta^T F)G \; , \tag{1.2}$$

where $d\mu$ is a functional measure, whose explicit form we will return to in great detail below. The partition function is then given by

$$\mathcal{Z}^X \; = \; \int d\mu \; e^{\frac{i}{\hbar}(W+X)} \; , \tag{1.3}$$

where both W and X satisfy the Quantum Master Equations

$$\Delta e^{\frac{i}{\hbar}W} \; = \; 0 \qquad \text{and} \qquad \Delta e^{\frac{i}{\hbar}X} \; = \; 0 \; , \tag{1.4}$$

respectively. The important observation is that W, which through the boundary conditions incorporates the classical action, and X, which does the required fixing of gauge symmetries, enter symmetrically. Viewing Δ as a generalized odd "Laplacian" which may potentially have quantum corrections that consist of higher order differential operators, we see that the integrand in (1.3) is required to be a product of two superharmonic functions. One may argue on general grounds that an arbitrary infinitesimal variation of the gauge-fixing part $e^{\frac{i}{\hbar}X}$ has the form

$$\delta e^{\frac{i}{\hbar}X} \; = \; [\overrightarrow{\Delta}, \delta\Psi]e^{\frac{i}{\hbar}X} \; , \tag{1.5}$$

or equivalently using (1.4), the variation δX is BRST-exact,

$$\delta X \; = \; \frac{\hbar}{i}e^{-\frac{i}{\hbar}X}\Delta(e^{\frac{i}{\hbar}X}\delta\Psi) \; \equiv \; \sigma_X(\delta\Psi) \; , \tag{1.6}$$

where σ_X is a quantum BRST-operator. Surprisingly, the independence of gauge-fixing X for the partition function $\mathcal{Z}^X$ can formally be demonstrated [7] by just using the above ingredients (1.1), (1.4) and (1.5), without reference to the detailed form of Δ:

$$\mathcal{Z}^{X+\delta X} - \mathcal{Z}^X \; = \; \int d\mu \; e^{\frac{i}{\hbar}W}[\overrightarrow{\Delta}, \delta\Psi]e^{\frac{i}{\hbar}X} \; = \; \int d\mu \left[e^{\frac{i}{\hbar}W}\delta\Psi \, (\Delta e^{\frac{i}{\hbar}X}) + (\Delta e^{\frac{i}{\hbar}W}) \, \delta\Psi \; e^{\frac{i}{\hbar}X}\right] \; = \; 0 \; . \tag{1.7}$$

The analogous statement (with X replaced by W) to show independence of the choice of gauge generating functional W was first studied by Tyutin and Voronov [8].

The fact that the most general description of gauge-fixing X puts it on equal footing with the construction of the action W means that there are situations in which gauge-fixing must require special attention. This happens when, be it for reasons of for example locality or unitarity, the gauge-fixing function X itself contains gauge degrees of freedom. It is of interest to clarify what happens in such a situation. What, in this formalism, fixes gauge symmetries of X? Remarkably, it turns out that the machinery is ready to tackle this more general situation, and provide the solution to the gauge-fixing problem. How this is achieved will be described in detail in this paper; the principle is simply that gauge symmetries of X are fixed by what used to play the rôle of only "action", W. We thus introduce

the notion of a *gauge-fixing* algebra in the X-part of the field-antifield formalism, very similar to the usual *gauge-generating* algebra inside the W-part. In order to give a very specific example where the more general quantization problem needs to be faced, we consider in Section 5 in detail the case where the gauge-fixing X is described in terms of a set of *reducible* gauge-fixing conditions G_α. The process may not stop there, since also the new gauge-fixings in W in turn may contain additional symmetries that need to be fixed as well. Then the formalism allows X to take over the gauge-fixing again, and so forth, for any finite number of steps in an alternating manner. In this way a multilevel construction is induced naturally. This is the subject of Section 6. We take the viewpoint that the existence of these new classes of theories must be taken seriously, and that their formal properties with respect to the quantization program therefore must be established.

The present paper is devoted to an exposition of this more general situation. However it is necessary first to establish a systematic and condensed formalism before approaching these new and interesting possibilities. We therefore begin in Section 2 with a discussion of some of the geometrical aspects of the field-antifield formalism from a covariant perspective. In particular we describe the properties of the measure density ρ, and the use of anticanonical transformations *i.e.* the antisymplectic analogue of canonical transformations. As an immediate application we extend the semi-density theory of Khudaverdian *et al.* [9] to the degenerate case. We next show how deformations of solutions to the Quantum Master Equation can be understood in terms of anticanonical transformations and associated measure function changes. This establishes a compact formula for the changes of gauge in the action, which is essential for all subsequent developments in the W-X multi-level formalism. While we shall give the necessary definitions below, we here just briefly remind the reader that the multi-level formalism must be introduced if one wishes to secure the most general and covariant construction that in particularly simple gauges reduce to the well-known field-antifield prescription that was presented in the original papers [1, 2]. In Section 3 we then turn to the situation where gauge-fixings in X are irreducible, exploring gauge-fixing at the first-level, and providing a new compact derivation of the form of X in that case. In Section 4 we return to the possibility of having antisymplectic second-class constraints in the path integral, a situation quite analogous to the more conventional case of symplectic second-class constraints with respect to the Poisson bracket in the Hamiltonian formalism. In particular, filling out a gap in the existing literature, we first establish a reduction theorem which explicitly demonstrates that the final gauge fixed path integral can be expressed, on a physical subspace of antisymplectic coordinates, in precisely the same form as the partition function (1.3). Secondly, we show that the second-class construction is manifestly invariant under reparametrizations of the second-class constraints, a vital investigation that taps into the very foundation of the antisymplectic Dirac construction. In Section 5 we consider a reducible gauge-fixing algebra and work out a general first-stage reducible theory in detail, and determining the associated path integral measure by solving the Master Equation. We perform several consistency checks by reduction techniques, linking reducible and irreducible descriptions of the gauge-fixing constraints, and comparing minimal and non-minimal approaches. Section 6 discusses the generalization of the first-level formalism to the above mentioned multi-level formalism. Finally, Section 7 contains our conclusions.

2 Antisymplectic Geometry Revisited

Let us start with a covariant, odd Δ-operator of second order,

$$\Delta = \Delta_\rho + V , \qquad\qquad \epsilon(\Delta) = 1 , \qquad\qquad (2.1)$$

$$\Delta_\rho \equiv \frac{(-1)^{\epsilon_A}}{2\rho} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} \rho E^{AB} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^B} , \qquad\qquad (2.2)$$

$$V = V^A \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} \ , \tag{2.3}$$

where Γ^A denotes local coordinates with Grassmann parity $\epsilon_A \equiv \epsilon(\Gamma^A)$. The Δ-operator is built with the help of covariant structure functions $E^{AB} = E^{AB}(\Gamma)$, $V^A = V^A(\Gamma)$, and $\rho = \rho(\Gamma)$ that transform under general coordinate transformations as a bi-vector, a vector and a density, respectively. We shall assume that E^{AB} has a Grassmann-graded skewsymmetry

$$E^{BA} = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)} E^{AB} \ . \tag{2.4}$$

Locally, eq. (2.1) describes the most general second-order odd Δ-operator such that

$$\Delta(1) = 0 \ . \tag{2.5}$$

The condition (2.5) is not vital for the construction below, but since currently there are no applications that would require $\Delta(1) \neq 0$, we shall not pursue such a possibility here. The antibracket of two functions $F = F(\Gamma)$ and $G = G(\Gamma)$ is defined via a double commutator* with the Δ-operator, acting on the constant function 1,

$$(F, G) \equiv -[[F, \overrightarrow{\Delta}], G]1 = (F \frac{\overleftarrow{\partial^r}}{\partial \Gamma^A}) E^{AB} (\frac{\overrightarrow{\partial^l}}{\partial \Gamma^B} G) \ , \tag{2.6}$$

where use was made of eq. (2.4). The square $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ is generally a third-order operator with no zero-order term $\Delta^2(1) = 0$. It becomes of second order if and only if a Grassmann-graded Jacobi identity

$$\sum_{F,G,H \text{ cycl.}} (-1)^{(\epsilon_F+1)(\epsilon_H+1)} (F, (G, H)) = 0 \tag{2.7}$$

for the antibracket holds. We shall assume this from now on. A bi-vector E^{AB} that satisfy skewsymmetry (2.4) and the Jacobi identity (2.7) is called a possibly degenerate *antisymplectic* bi-vector. There is an antisymplectic analogue of Darboux's Theorem that states that locally, if the rank of E^{AB} is constant, there exist Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi_\alpha^*; \Theta^a\}$, such that the only non-vanishing antibrackets between the coordinates are $(\phi^\alpha, \phi_\beta^*) = \delta_\beta^\alpha = -(\phi_\beta^*, \phi^\alpha)$. In other words, the Jacobi identity is the integrability condition for the Darboux coordinates. The variables ϕ^α, ϕ_α^* and Θ^a are called *fields*, *antifields* and *Casimirs*, respectively. Granted the Jacobi identity (2.7), the square of the Δ-operator is a first order differential operator,

$$\Delta^2(FG) = \Delta^2(F) G + F \Delta^2(G) \ , \tag{2.8}$$

if and only if there is a Leibniz rule for the interplay of Δ and the antibracket

$$\Delta(F, G) = (\Delta(F), G) - (-1)^{\epsilon_F}(F, \Delta(G)) \ . \tag{2.9}$$

We shall also assume this to be the case. It is interesting to note that the Leibniz rule (2.9) holds automatically for a conventional odd Laplacian Δ_ρ (still assuming the Jacobi identity (2.7)), so the Leibniz rule (2.9) actually reduces to

$$V(F, G) = (V(F), G) - (-1)^{\epsilon_F}(F, V(G)) \ . \tag{2.10}$$

We see that V is a generating vector field for an anticanonical transformation.

In the degenerate case, one would usually proceed by investigating an antisymplectic leaf/orbit where the values of the Casimirs Θ^a are kept fixed. Then seen from within such leaf the antisymplectic

---

*Here, and throughout the paper, $[A, B]$ denotes the graded commutator $[A, B] = AB - (-1)^{\epsilon_A \epsilon_B} BA$.

structure will appear non-degenerate. An example of this is the case of antisymplectic second-class constraints, which will be the subject of Section 4. On the other hand, if E^{AB} is non-degenerate, the V in eq. (2.10) becomes locally an Hamiltonian vector field, *i.e.* there exists a bosonic Hamiltonian H such that $V = (H, \cdot)$. It follows that one can locally absorb the V-term into a rescaling of the measure density $\rho \to \rho' = e^{2H}\rho$. Since ρ and H are intimately related through this mechanism one may regard the Leibniz rule (2.9) as an integrability condition for the local existence of ρ. In any case, we shall from now on only consider the conventional odd Laplacian Δ_ρ without the V-term.

2.1 Compatible Structures

A measure density ρ and a possibly degenerate antisymplectic E^{AB} are called *compatible* if and only if the odd Laplacian Δ_ρ is nilpotent,

$$\Delta_\rho^2 = 0 . \tag{2.11}$$

Constant ρ and constant E^{AB} are the most important example of compatible structures. It is interesting to classify the compatible structures within the set of all pairs (ρ, E). To this end, consider two Δ-operators sharing the same antisymplectic structure E, and with two different measure densities ρ and ρ', respectively, that are not necessarily compatible with E. They differ by a Hamiltonian vector field,

$$\Delta_{\rho'} - \Delta_\rho = (\ln\sqrt{\frac{\rho'}{\rho}}, \ \cdot \ ) . \tag{2.12}$$

Also the difference in their squares is a Hamiltonian vector field [10, 11]:

$$\Delta_{\rho'}^2 - \Delta_\rho^2 = (\nu(\rho'; \rho, E), \ \cdot \ ) . \tag{2.13}$$

Here we have introduce a Grassmann-odd function

$$\nu(\rho'; \rho, E) \equiv \sqrt{\frac{\rho}{\rho'}}(\Delta_\rho\sqrt{\frac{\rho'}{\rho}}) = \frac{1}{\sqrt{\rho'}}(\Delta_1\sqrt{\rho'}) - \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho}) \tag{2.14}$$

of a measure density ρ' with respect to a reference system (ρ, E). The quantity ν acts as a scalar under general coordinate transformations, and satisfies the following 2-cocycle condition [9]:

$$\nu(\rho_1; \rho_2, E) + \nu(\rho_2; \rho_3, E) + \nu(\rho_3; \rho_1, E) = 0 , \tag{2.15}$$

and, as trivial consequences thereof,

$$\begin{aligned} \nu(\rho_1; \rho_1, E) &= 0 , \\ \nu(\rho_1; \rho_2, E) + \nu(\rho_2; \rho_1, E) &= 0 . \end{aligned} \tag{2.16}$$

In fact, $\nu(\rho'; \rho, E)$ can be written globally as a difference of a scalar function $\nu(\rho; E)$,

$$\nu(\rho'; \rho, E) = \nu(\rho'; E) - \nu(\rho; E) . \tag{2.17}$$

To see eq. (2.17), go to a coordinate system where E^{AB} becomes equal to a constant antisymplectic reference matrix E_0^{AB}, which we for simplicity take to be the Darboux matrix. It follows from the antisymplectic analogue of Darboux's Theorem that one may cover the manifold with such coordinate charts, except for singular points where the rank of E jumps. In this Section we shall denote a Δ-operator corresponding to constant $\rho = 1$ and $E^{AB} = E_0^{AB}$ as Δ_0. Now define

$$\nu(\rho; E_0) \equiv \nu(\rho; 1, E_0) = \frac{1}{\sqrt{\rho}}(\Delta_0\sqrt{\rho}) , \tag{2.18}$$

5

where the arguments ρ, 1 and E_0 all refer to the above Darboux coordinate system. For the definition (2.18) to be well-defined, one should justify that two different choices of Darboux coordinates lead to same value of ν. By definition, any two Darboux coordinate systems are connected by an anticanonical transformation. According to Lemma 2.1 below the Jacobian

$$J_{fi} \equiv \mathrm{sdet}\left(\frac{\partial \Gamma_f^A}{\partial \Gamma_i^B}\right) \tag{2.19}$$

associated to an anticanonical transformation $\Gamma_i^A \to \Gamma_f^A$ has a vanishing ν:

$$\nu(J_{fi}; E_0) \equiv \nu(J_{fi}; 1, E_0) = \frac{1}{\sqrt{J_{fi}}}(\Delta_0\sqrt{J_{fi}}) = 0 . \tag{2.20}$$

Hence it follows from the 2-cocycle condition (2.15) that the ν-definition (2.18) does not depend on the particular choice of Darboux coordinate system:

$$\nu(\rho_f; 1, E_0) = \nu(\frac{\rho_i}{J_{fi}}; 1, E_0) = \nu(\rho_i; J_{fi}, E_0) = \nu(\rho_i; 1, E_0) - \nu(J_{fi}; 1, E_0) = \nu(\rho_i; 1, E_0) . \tag{2.21}$$

In this way one achieves a well-defined function $\nu(\rho; E_0)$ on the set of all Darboux coordinate charts. It is assumed that the definition can be extended uniquely to singular points by continuity. One generalizes the definition of $\nu(\rho; E)$ to an arbitrary coordinate system Γ^A by requiring that $\nu(\rho; E)$ is a scalar under general coordinate transformations, $i.e.$

$$\nu(\rho; E) \equiv \nu(\frac{\rho}{J}; E_0) , \tag{2.22}$$

where $J \equiv \mathrm{sdet}\left(\frac{\partial \Gamma_0}{\partial \Gamma}\right)$ denotes the Jacobian of a transformation $\Gamma^A \to \Gamma_0^A$ into some Darboux coordinate system Γ_0^A. One may easily check that this definition fulfills eq. (2.17). Moreover, the definition is independent of the constant reference matrix E_0^{AB}. We shall from now on use a shorthand notation $\nu_\rho \equiv \nu(\rho; E)$. Next define an operator Δ_E that takes semi-densities to semi-densities [9]

$$\Delta_E(\sqrt{\rho}) \equiv \sqrt{\rho}\,\nu_\rho , \tag{2.23}$$

$i.e.$ for Darboux coordinates it is simply

$$\Delta_E(\sqrt{\rho}) \equiv \Delta_0(\sqrt{\rho}) . \tag{2.24}$$

We emphasize that the constructions of ν_ρ and Δ_E rely heavily on Lemma 2.1. The operator Δ_E is nilpotent,

$$\Delta_E^2 = 0 . \tag{2.25}$$

Eq. (2.25) encodes precisely the antisymplectic data (2.4) and (2.7) without information about any particular ρ. On the other hand, the odd Laplacian Δ_ρ, which takes scalars to scalars, consists of both the E-structure in Δ_E and a measure density ρ,

$$\Delta_\rho(F) = \frac{1}{\sqrt{\rho}}[\vec{\Delta}_E, F]\sqrt{\rho} = (\nu_{F^2\rho} - \nu_\rho)F , \qquad \epsilon(F) = 0 . \tag{2.26}$$

The nilpotency of Δ_E implies

$$(\Delta_\rho + \nu_\rho)^2 = 0 , \tag{2.27}$$

$$(\Delta_\rho \nu_\rho) = 0 , \tag{2.28}$$

$$\Delta_\rho^2 = (\nu_\rho , \,\cdot\,) . \tag{2.29}$$

We summarize the above information in Fig. 1.

6

Figure 1: The following diagram holds for an arbitrary pair of measure density ρ and possibly degenerate antisymplectic structure E, cf. [9]:

$$
\begin{array}{ccccc}
\begin{array}{c} \exists\ \text{Darboux coordinate} \\ \text{system such that } \rho = 1. \end{array} & \Leftrightarrow & & & \begin{array}{c} \exists\ \text{Darboux coordinate system and} \\ \exists\ \text{anticanonical transformation} \\ \text{such that } \rho = J, \text{ the Jacobian.} \end{array} \\
& & & & \Downarrow \\
\Delta_\rho^2 = 0 \quad \Leftrightarrow \quad \nu_\rho \text{ is a Casimir} & & \Leftarrow & & \nu_\rho = 0 \\
& \Updownarrow & & & \\
& \begin{array}{c} \exists\ \text{Darboux coordinate system} \\ \text{such that } \sqrt{\rho} \text{ eigenvector} \\ \text{for } \Delta_0 \text{ with eigenvalue } \nu_\rho. \end{array} & & &
\end{array}
\tag{2.30}
$$

The above eigenvalue is constant within an antisymplectic leaf/orbit. In the non-degenerate case the Casimir ν_ρ is a Grassmann-odd constant. Evidently Grassmann-odd constants cannot be non-zero, if the theory does not have any external Grassmann-odd parameters. In practice, this is the case. The diagram contains two implication arrows $\Downarrow$, that are not bi-implications $\Updownarrow$. One is the possibility of a non-zero Casimir ν_ρ; the other is a non-trivial Δ-cohomology obstruction, cf. Subsection 2.3.

2.2 Anticanonical Transformations

An anticanonical transformation preserves by definition the antisymplectic structure E. Infinitesimally, it is generated by a bosonic vector field X such that

$$
X(F, G) = (X(F), G) + (F, X(G)) .
\tag{2.31}
$$

A Hamiltonian vector field $X = \mathrm{ad}\Psi$, where $\mathrm{ad}\Psi \equiv (\Psi, \cdot)$ denotes the "adjoint action" with respect to the antibracket, and where Ψ is a Grassmann-odd generator, is an example of an infinitesimal anticanonical transformations (2.31). This follows directly from the Jacobi identity (2.7). It is natural to call an infinitesimal anticanonical transformation X in eq. (2.31) for ad-*closed*, and a Hamiltonian vector field $X = \mathrm{ad}\Psi$ for ad-*exact*. If E is non-degenerate, then all ad-closed vector fields are locally of the ad-exact type. Here we shall elaborate on ad-closed vector fields in a possibly degenerate antisymplectic manifold. To this end, let

$$
\mathrm{div}_\rho X \equiv \frac{(-1)^{\epsilon_A}}{\rho} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} (\rho X^A)
\tag{2.32}
$$

denote the divergence of a vector field X with respect to a measure density ρ. Then

$$
X \text{ ad}-\text{closed} \quad \Rightarrow \quad [\overrightarrow{\Delta}_\rho, X] = -\frac{1}{2}(\mathrm{div}_\rho X, \ \cdot \ ) ,
\tag{2.33}
$$

and

$$
X \text{ ad}-\text{closed} \quad \Rightarrow \quad \frac{1}{2}(\Delta_\rho \, \mathrm{div}_\rho X) = X(\nu_\rho) .
\tag{2.34}
$$

Equations (2.33) and (2.34) are ad-closed versions of the Leibniz rule (2.9) and the relation (2.29), respectively. They reduce to those relations, if X is ad-exact, because the odd Laplacian is the divergence of a Hamiltonian vector field [10, 11]

$$\Delta_\rho \Psi \;=\; -\frac{1}{2}\mathrm{div}_\rho(\mathrm{ad}\Psi)\;, \qquad\qquad \epsilon(\Psi)\;=\;1\;. \qquad\qquad (2.35)$$

In Darboux coordinates with $\rho=1$ the eq. (2.34) becomes

$$X\ \mathrm{ad-closed} \qquad \Rightarrow \qquad (\Delta_1\ \mathrm{div}_1 X)\;=\;0\;. \qquad\qquad (2.36)$$

This non-covariant result will be needed for the Lemma 2.1 below, which in turn is used to justify the definition (2.18) of ν_ρ. To avoid circular logic, we mention that the special case eq. (2.36) can also be proven directly without relying on the concept of ν_ρ.

Consider now a one-parameter family of (not necessarily anticanonical) passive coordinate transformations $\Gamma^A(t)$ for some parameter $t \in [t_i, t_f]$, and governed by a one-parameter generating vector field $X_t = X_t(\Gamma)$,

$$\frac{d\Gamma^A(t)}{dt} \;=\; X_t^A\;. \qquad\qquad (2.37)$$

We are here and below guilty of infusing some active picture language into a passive picture, *i.e.* properly speaking, the active vector field is *minus* X, and so forth. The solution

$$\Gamma^A(t) \;=\; U(t;t_i)\Gamma^A(t_i) \qquad\qquad (2.38)$$

can be expressed with the help of a path-ordered exponential

$$U(t_f;t_i) \;\equiv\; \mathcal{P}\exp\int_{t_i}^{t_f} dt\ X_t\;. \qquad\qquad (2.39)$$

The Jacobian

$$J(t_f;t_i) \;\equiv\; \mathrm{sdet}\left(\frac{\partial\Gamma^A(t_f)}{\partial\Gamma^B(t_i)}\right) \qquad\qquad (2.40)$$

is given by

$$\ln J(t_f;t_i) \;=\; \int_{t_i}^{t_f} dt\ U(t_f;t)\mathrm{div}_1^{(t_i)}X_t\;, \qquad\qquad (2.41)$$

where $\mathrm{div}_1^{(t)}$ refers to the divergence with $\rho=1$ in the coordinates $\Gamma^A(t)$. The formula (2.41) can be deduced from the differential equation

$$\frac{d}{dt}\ln J(t;t_i) \;=\; (-1)^{\epsilon_A}\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A(t)}\frac{d\Gamma^A(t)}{dt} \;=\; \mathrm{div}_1^{(t)}X_t \;=\; \mathrm{div}_1^{(t_i)}X_t + X_t\left(\ln J(t;t_i)\right)\;. \qquad (2.42)$$

The measure density ρ transforms in the passive picture with the Jacobian

$$\rho(t_f) \;=\; \frac{\rho(t_i)}{J(t_f;t_i)}\;. \qquad\qquad (2.43)$$

Therefore ρ satisfies the following differential equation

$$\frac{d}{dt}\ln\rho(t) \;=\; -\,\mathrm{div}_1^{(t)}X_t \;=\; X_t(\ln\rho(t)) - \mathrm{div}_\rho X_t\;. \qquad\qquad (2.44)$$

Next put ρ back into the divergence in eq. (2.41). Then

$$\frac{J(t_f;t_i)}{\rho(t_i)}U(t_f;t_i)\rho(t_i) \;=\; \exp\left[\int_{t_i}^{t_f} dt\ U(t_f;t)\mathrm{div}_\rho X_t\right]\;. \qquad\qquad (2.45)$$

8

This equation can be understood both in the active and the passive picture, and it will play a key rôle later on. In deriving it we have used that

$$\int_{t_i}^{t_f} dt \; U(t_f;t) X_t \left(\ln \rho(t_i) \right) \;\; = \;\; - \int_{t_i}^{t_f} dt \; \frac{d}{dt} \left[U(t_f;t) \ln \rho(t_i) \right] \;\; = \;\; \left[U(t_f;t_i) - \mathbf{1} \right] \ln \rho(t_i) \; . \qquad (2.46)$$

Finally, recall the remarkable fact that the Δ-operator – despite being a second order differential operator – has a Leibniz type interplay with ad-closed vector fields, cf. eq. (2.33). For this reason we can form identities that look very similar to well-known identities in ordinary Quantum Mechanics. For instance,

$$X_t \; \mathrm{ad-closed} \qquad \Rightarrow \qquad [\vec{\Delta}_\rho, U(t_f;t_i)] \;\; = \;\; -\frac{1}{2} \int_{t_i}^{t_f} dt \; U(t_f;t) \; \mathrm{ad}(\mathrm{div}_\rho X_t) \; U(t;t_i) \; . \qquad (2.47)$$

We may now give a short direct proof of a result used in eq. (2.20):

Lemma 2.1 *Let $\Gamma^A(t_i) \to \Gamma^A(t_f)$ be a finite anticanonical transformation between Darboux coordinates in a possibly degenerate antisymplectic manifold. Then the Jacobian $J(t_f;t_i)$ satisfies*

$$\Delta_1^{(t_i)} \sqrt{J(t_f;t_i)} \;\; = \;\; 0 \; . \qquad (2.48)$$

Here $\Delta_1^{(t_i)}$ refers to the odd Laplacian with $\rho{=}1$ in the initial Darboux coordinates $\Gamma^A(t_i)$.

PROOF OF LEMMA 2.1:

$$\begin{aligned}
-\Delta_1^{(t_i)} \ln \sqrt{J(t_f;t_i)} \;\; &= \;\; -\frac{1}{2} \int_{t_i}^{t_f} dt \; [\Delta_1^{(t_i)}, U(t_f;t)] \mathrm{div}_1^{(t_i)} X_t \\
&= \;\; \frac{1}{4} \int_{t_i}^{t_f} dt \int_t^{t_f} dt' \; U(t_f;t') \left(\mathrm{div}_1^{(t_i)} X_{t'}, \; U(t';t) \mathrm{div}_1^{(t_i)} X_t \right) \\
&= \;\; \frac{1}{4} \left(\int_{t_i}^{t_f} dt' \; U(t_f;t') \mathrm{div}_1^{(t_i)} X_{t'}, \int_{t_i}^{t_f} dt \; U(t_f;t) \mathrm{div}_1^{(t_i)} X_t \right) \; \theta(t'{-}t) \\
&= \;\; \frac{1}{2} \left(\ln \sqrt{J(t_f;t_i)}, \; \ln \sqrt{J(t_f;t_i)} \right) \; , \qquad (2.49)
\end{aligned}$$

where use has been made of the equations (2.36), (2.41) and (2.47).

$$\square$$

Lemma 2.1 is a degenerate generalization of a well-known result [3, 9] for the non-degenerate case. The covariant version of Lemma 2.1 reads

$$\left(\frac{U(t_f;t_i)\rho(t_i)}{\rho(t_f)} \right)^{-\frac{1}{2}} \Delta_\rho \left(\frac{U(t_f;t_i)\rho(t_i)}{\rho(t_f)} \right)^{\frac{1}{2}} \;\; = \;\; \nu \left(U(t_f;t_i)\rho(t_i); \; \rho(t_f), E(t_f) \right) \;\; = \;\; \left[U(t_f;t_i) - \mathbf{1} \right] \nu_\rho \; , \qquad (2.50)$$

where $\Gamma^A(t_i) \to \Gamma^A(t_f) \equiv U(t_f;t_i)\Gamma^A(t_i)$ is a finite anticanonical transformation.

We observe that it was never necessary in this Subsection to assume that ρ and E are compatible, *i.e.* that Δ_ρ is nilpotent. In the remaining part of the paper we shall assume that the Δ-operator is nilpotent, except for a subtlety (4.36) concerning second-class constraints.

2.3 Varying the Solutions to the Quantum Master Equation

Let us now consider solutions to the Quantum Master Equation.

- We are mainly interested in deformations of the quantum master action W in $\Delta_\rho e^{\frac{i}{\hbar}W} = 0$ for nilpotent Δ_ρ, where W satisfies certain rank and boundary conditions [1, 2, 3]. Here both ρ and E are kept fixed.

- As a precursor for the above problem, it is of interest to vary the semi-density $\sqrt{\rho}$ in $\Delta_1^{(t_i)}\sqrt{\rho} = 0$ for nilpotent $\Delta_1^{(t_i)}$. Here the antisymplectic structure E is kept fixed.

Let us collectively write $(\Delta_\rho \sigma) = 0$ to represent both types of problems, so that $\sigma \in \Sigma$ denotes a solution in the space Σ of solutions to the Quantum Master Equation. Now consider a one-parameter family of solutions $\sigma(t)$, where $t \in [t_i, t_f]$. Obviously, the difference of neighboring solutions is Δ-closed: $\Delta_\rho(d\sigma/dt) = 0$. If the difference is furthermore Δ-exact, we may write[†]

$$\frac{d\sigma(t)}{dt} \;=\; -[\vec{\Delta}_\rho, \Psi(t)]\sigma(t) \;, \tag{2.51}$$

where $\Psi(t)$ is a one-parameter family of fermionic functions. Next introduce a path-ordered exponential

$$V(t_f; t_i) \;\equiv\; \mathcal{P}\exp\left[-\int_{t_i}^{t_f} dt\; [\vec{\Delta}_\rho, \Psi(t)]\right] \;. \tag{2.52}$$

Integrating eq. (2.51) along the path, we find

$$\sigma(t_f) \;=\; V(t_f; t_i)\sigma(t_i) \;=\; V(t_f; t_i)\sigma(t_i)V(t_i; t_f)V(t_f; t_i)1 \;=\; U(t_f; t_i)\sigma(t_i) \cdot v(t_f; t_i) \;, \tag{2.53}$$

where we have used the identity

$$e^{-[\vec{\Delta},\Psi]}\sigma e^{[\vec{\Delta},\Psi]} \;=\; e^{-[[\vec{\Delta},\Psi],\,\cdot\,]}\sigma \;=\; e^{\mathrm{ad}\psi}\sigma \;, \tag{2.54}$$

and defined

$$v(t_f; t_i) \;\equiv\; V(t_f; t_i)1 \;=\; \exp\left[-\int_{t_i}^{t_f} dt\; U(t_f; t)\Delta_\rho\Psi(t).\right] \;=\; \sqrt{\frac{J(t_f; t_i)}{\rho(t_i)}}U(t_f; t_i)\rho(t_i) \;. \tag{2.55}$$

Use has been made of eq. (2.35), eq. (2.45) and the differential equation

$$\frac{d}{dt}v(t; t_i) \;=\; -[\vec{\Delta}_\rho, \Psi(t)]v(t; t_i) \;=\; (\Psi(t), v(t; t_i)) - (\Delta_\rho\Psi(t))v(t; t_i) \;. \tag{2.56}$$

In the third equality of (2.55) we re-interpret the $\Psi(t)$ family, which originates from a Δ-exact variation, as a generator of an ad-exact anticanonical transformation. There is thus a one-to-one correspondence between ad-exact anticanonical transformations and Δ-exact variations. Moreover, as anticanonical transformations can be understood passively, one may also give Δ-exact variations a passive interpretation, *i.e.* one is not changing the solution σ; only the coordinates Γ^A. The detailed mechanism for this one-to-one correspondence is of great interest, both conceptionally and in practice.

[†]In general, there is non-trivial Δ-cohomology. In finite dimensions, for a constant non-degenerate antisymplectic matrix E^{AB}, whose fermionic blocks vanish, the non-trivial Δ_0-cohomology is one-dimensional, generated by the *fermionic top-monomial*, *i.e.* the monomial of all fermionic and no bosonic variables [12]. In local field theory, cohomology may arise from locality requirements. Furthermore, our treatment obviously only applies to a path-connected solution space.

Definition 2.2 *We shall say that an anticanonical transformation $U(t_f; t_i)$ acts on a pair $\sigma(t_i)$ and $\rho(t_i)$ according to the following "twisted" transformation rules:*

$$\sigma(t_i) \ \longrightarrow \ \sigma(t_f) \ = \ \sqrt{\frac{J(t_f; t_i)}{\rho(t_i)}} U(t_f; t_i) \left[\sigma(t_i) \sqrt{\rho(t_i)} \right] \ , \tag{2.57}$$

$$\rho(t_i) \ \longrightarrow \ \rho(t_f) \ = \ \frac{\rho(t_i)}{J(t_f; t_i)} \ . \tag{2.58}$$

To be more precise, it is the stabilizer subgroup $\{U \in \mathcal{G} \mid U(\nu_\rho) = \nu_\rho\}$ that acts on solutions to the Quantum Master Equation $\Delta_\rho \sigma = 0$. The full group $\mathcal{G}$ of anticanonical transformations acts on solutions to the modified Quantum Master Equation $(\Delta_\rho + \nu_\rho)\sigma = 0$.

Letting the anticanonical generator $\Psi(t)$ depend on t is somewhat academic, because one may always find an equivalent constant generator Ψ (and choose the parameter interval to be $[t_i, t_f] = [0, 1]$), such that $U(t_f = 1; t_i = 0) = e^{\mathrm{ad}\Psi}$. While t-dependent Ψ's provide a deeper theoretical understanding, it is preferred in practice to work with such t-independent Ψ's where path-ordering issues are absent. In the latter case the above one-parameter solution is of the form

$$\sigma(t) \ = \ e^{-t[\vec{\Delta}, \Psi]} \sigma_i \ , \tag{2.59}$$

and eq. (2.45) reduces to

$$\ln \sqrt{\frac{J(t_f; t_i)}{\rho(t_i)}} U(t_f; t_i) \rho(t_i) \ = \ - E(\mathrm{ad}\Psi) \Delta_\rho \Psi \ , \tag{2.60}$$

where

$$E(x) \ = \ \int_{t_i=0}^{t_f=1} dt \ e^{xt} \ = \ \frac{e^x - 1}{x} \ . \tag{2.61}$$

It follows from eqs. (2.53) and (2.55) that

Proposition 2.3 : *A finite Δ-exact transformation*

$$e^{\frac{i}{\hbar} W_f} \ = \ e^{-[\vec{\Delta}, \Psi]} e^{\frac{i}{\hbar} W_i} \tag{2.62}$$

deforms the quantum action W according to

$$W_f \ = \ e^{\mathrm{ad}\Psi} W_i + (i\hbar) E(\mathrm{ad}\Psi) \Delta \Psi \ = \ e^{\mathrm{ad}\Psi} W_i + (i\hbar) \frac{e^{\mathrm{ad}\Psi} - 1}{\mathrm{ad}\Psi} \Delta \Psi \ . \tag{2.63}$$

This important deformation formula will be used repeatedly throughout the remainder of the paper. By expanding in Planck's constant,

$$W \ = \ S + \sum_{n=1}^{\infty} (i\hbar)^n W_n \ , \qquad\qquad \Psi \ = \ \sum_{n=0}^{\infty} (i\hbar)^n \Psi_n \ , \tag{2.64}$$

one sees that the classical action S undergoes a classical anticanonical transformation

$$S_f \ = \ e^{\mathrm{ad}\Psi_0} S_i \ , \tag{2.65}$$

11

while the leading quantum correction W_1 transforms as

$$W_{1,f} \;=\; e^{\mathrm{ad}\Psi_0}W_{1,i} + E(\mathrm{ad}\Psi_0)\Delta\Psi_0 + (E(\mathrm{ad}\Psi_0)\Psi_1, S_f) \;. \tag{2.66}$$

To summarize, the deformations of the *classical* solutions S to the Classical Master Equation $(S,S)=0$ are generated by the group of anticanonical transformations $e^{\mathrm{ad}\Psi_0}$, cf. (2.65). This should be compared to the *quantum* situation, where deformations of solutions W to the Quantum Master Equation $\Delta e^{\frac{i}{\hbar}W}=0$ are similarly generated by the group of quantum anticanonical transformations $e^{\mathrm{ad}\Psi}$. Here Ψ depends on $\hbar$ in accordance with (2.64), but with the important difference that the group action is applied in a non-standard way, twisted by the semi-density $\sqrt{\rho}$, cf. eq. (2.57). This twisting effect is not felt at the classical level.

There are important exceptions where the above twisting is not present at all. This happens for instance in Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ when $\rho = \rho(\phi)$ and $\Psi = \Psi(\phi)$ are independent of the antifields ϕ^*_α, so that $\Delta\Psi = 0$. Then the formula (2.63) reduces to a purely anticanonical transformation

$$W_f \;=\; e^{\mathrm{ad}\Psi}W_i \;=\; W_i(\phi; \phi^*+\frac{\partial\Psi}{\partial\phi}) \;, \tag{2.67}$$

a formula that is intimately tied to the original way of gauge-fixing in the field-antifield formalism [1].

The dilation transformation $e^{\frac{i}{\hbar}W} \to Ce^{\frac{i}{\hbar}W}$ (or $e^{\frac{i}{\hbar}X} \to Ce^{\frac{i}{\hbar}X}$), where C is a constant factor, is clearly a symmetry of the Quantum Master Equation. From a mathematical standpoint, granted that the action satisfies pertinent rank conditions, the scaling represents non-trivial Δ-cohomology, which is excluded from our reasoning (1.5) of gauge independence for $\mathcal{Z}$. It obviously does change the partition function $\mathcal{Z} \to C\mathcal{Z}$. On the other hand, from a physics perspective such an overall constant rescaling is totally trivial and plays no rôle whatsoever.

As another simple application, let us briefly mention the second type of problem. We have a nilpotent Δ-operator $\Delta_1^{(t_i)}$ with $\rho(t_i)=1$ in the coordinates $\Gamma^A(t_i)$. The constant semi-density $\sigma(t_i)=1$ is a trivial solution to $(\Delta_1^{(t_i)}\sigma) = 0$. Now act with an anticanonical transformation $U(t_f;t_i)$ on $\sigma(t_i)=1$ according to the transformation rule eq. (2.57). Then

$$\rho(t_f) \;=\; \frac{\rho(t_i)}{J(t_f;t_i)} \;=\; \frac{1}{J(t_f;t_i)} \;, \tag{2.68}$$

so the transformed semi-density becomes

$$\sigma(t_f) \;=\; \frac{U(t_f;t_i)\left[\sigma(t_i)\sqrt{\rho(t_i)}\right]}{\sqrt{\rho(t_f)}} \;=\; \sqrt{J(t_f;t_i)} \;. \tag{2.69}$$

This in turn provides a descriptive proof of Lemma 2.1.

3 Irreducible First-Level Gauge-Fixing Formalism

3.1 Review of Original Gauge-Fixing Formalism

Our starting point is an action $W = W(\Gamma;\hbar)$ that possesses N gauge symmetries that should be fixed. In the original gauge-fixing prescription of the field-antifield formalism [1, 2, 3] there is no X-part. In non-degenerate Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ with a density ρ that is independent of the antifields ϕ^*_α, we may reformulate gauge-fixing in the following way that is easy to generalize later:

1a. First change the Boltzmann factor $e^{\frac{i}{\hbar}W}$ with a Δ-exact transformation generated by a gauge fermion function Ψ,

$$e^{\frac{i}{\hbar}W^{\Psi}} \;\; = \;\; e^{-[\vec{\Delta},\Psi]}e^{\frac{i}{\hbar}W} \; . \tag{3.1}$$

This formula obviously preserves the Quantum Master Equation, and it generalizes readily to a gauge fermion *operator* $\hat{\Psi}$, but we shall not pursue such a generalization here. According to eq. (2.63) the transformed action W^{Ψ} becomes

$$W^{\Psi} \;\; = \;\; e^{\mathrm{ad}\Psi}W + (i\hbar)E(\mathrm{ad}\Psi)\Delta\Psi \; . \tag{3.2}$$

1b. Next put the antifields $\phi_{\alpha}^{*} \to 0$ to zero,

$$\mathcal{Z}^{\Psi} \;\; = \;\; \int[d\phi]\,\rho\,e^{\frac{i}{\hbar}W^{\Psi}}\Big|_{\phi^{*}=0} \;\; = \;\; \int[d\Gamma]\,\rho\,\delta(\phi^{*})\,e^{\frac{i}{\hbar}W^{\Psi}} \; . \tag{3.3}$$

1c. The partition function $\mathcal{Z}^{\Psi}$ defined this way does not depend on the gauge fermion Ψ. PROOF: Start with exponentiating the δ-function in eq. (3.3):

$$\delta(\phi^{*}) \;\; = \;\; \int[d\lambda]e^{\frac{i}{\hbar}X} \tag{3.4}$$

with a trivial action $X = \phi_{\alpha}^{*}\lambda^{\alpha}$ that obviously satisfies the Quantum Master Equation $\Delta_{[1]}e^{\frac{i}{\hbar}X} = 0$, where $\Delta_{[1]} \equiv \Delta + (-1)^{\epsilon_{\alpha}}(\partial/\partial\lambda^{\alpha})(\partial/\partial\lambda_{\alpha}^{*})$ is the suitably extended Δ-operator. Then the partition function

$$\mathcal{Z}^{\Psi} = \int[d\Gamma][d\lambda]\rho\,e^{\frac{i}{\hbar}X}e^{-[\vec{\Delta}_{[1]},\Psi]}e^{\frac{i}{\hbar}W} \tag{3.5}$$

becomes of the W-X–form discussed in the Introduction. The independence of Ψ follows straightforwardly from the symmetry (1.1) of the Δ-operator,

$$\mathcal{Z}^{\Psi+\delta\Psi} - \mathcal{Z}^{\Psi} \;\; = \;\; \int[d\Gamma][d\lambda]\rho\int_{0}^{1}dt\,e^{-(1-t)[\vec{\Delta}_{[1]},\Psi]}e^{\frac{i}{\hbar}W}\,[\vec{\Delta}_{[1]},\delta\Psi]e^{t[\vec{\Delta}_{[1]},\Psi]}e^{\frac{i}{\hbar}X} \;\; = \;\; 0 \; . \tag{3.6}$$

The λ^{α}'s and the $\Delta_{[1]}$ can be viewed as part of the so-called first-level formalism, cf. Subsection 3.2.

$\square$

2a. One may reach an alternative version of the partition function (3.3) by using the symmetry (1.1) of the Δ-operator to write eq. (3.3) as

$$\mathcal{Z}^{\Psi} \;\; = \;\; \int[d\Gamma]\,\rho\,e^{\frac{i}{\hbar}W}e^{[\vec{\Delta},\Psi]}\delta(\phi^{*}) \;\; = \;\; \int[d\Gamma]\,\rho\,e^{\frac{i}{\hbar}W}\delta\left(e^{-\mathrm{ad}\Psi}\phi^{*}\right)e^{E(-\mathrm{ad}\Psi)\Delta\Psi} \; . \tag{3.7}$$

2b. If furthermore the gauge fermion Ψ is independent of the antifields ϕ_{α}^{*}, as is normally assumed, the transformation (3.2) reduces to a purely anticanonial transformation,

$$W^{\Psi} \;\; = \;\; e^{\mathrm{ad}\Psi}W \; , \tag{3.8}$$

and one arrives at the familiar prescription of the original field-antifield formalism, where gauge-fixing is done by an explicit substitution of the antifields $\phi_{\alpha}^{*} \to \partial\Psi/\partial\phi^{\alpha}$ with a field gradient of a gauge fermion Ψ:

$$\mathcal{Z}^{\Psi} \;\; = \;\; \int[d\phi]\,\rho\,e^{\frac{i}{\hbar}W}\Big|_{\phi^{*}=\frac{\partial\Psi}{\partial\phi}} \; . \tag{3.9}$$

The above gauge-fixing procedure with explicit removal of the N antifields obviously refers to a particular set of coordinates on the supermanifold, and is therefore not covariant. The X-part of the new formulation is precisely introduced [6] as a covariantization of the gauge-fixing prescription.

3.2 First-Level Formalism

The gauge-fixing procedure was considerably generalized in the nineties into a so-called *multi-level formalism* [4, 5, 6] to allow for covariant and more flexible gauge-fixing choices. For systematic reasons we shall retroactively call the original non-degenerate antisymplectic phase space variables $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ for *zeroth-level fields*, the expansion parameter $\hbar \equiv \hbar_{(0)}$ for the *zeroth-level Planck constant*, and the gauge-fixing procedure of the last subsection for *zeroth-level gauge-fixing*.

In the (irreducible) *first-level* formalism one introduces N Lagrange multipliers $\lambda^\alpha \equiv \lambda^\alpha_{(1)}$ of Grassmann parity $\epsilon_\alpha \equiv \epsilon^{(1)}_\alpha$ and N antifields $\lambda^*_\alpha \equiv \lambda^*_{(1)\alpha}$, which we collectively call the *first-level fields*. The phase space variables

$$\Gamma^A_{[1]} \;\equiv\; \left\{ \Gamma^A; \lambda^\alpha, \lambda^*_\alpha \right\} \tag{3.10}$$

for the first-level formalism thus consist of the zeroth and the first-level fields[‡]. The first-level odd Laplacian

$$\Delta_{[1]} \;\equiv\; \Delta + (-1)^{\epsilon_\alpha} \frac{\overrightarrow{\partial^l}}{\partial\lambda^\alpha} \frac{\overrightarrow{\partial^l}}{\partial\lambda^*_\alpha} \tag{3.11}$$

gives rise to an extended antisymplectic structure in the standard way. We shall always assume there is a trivial measure density associated with the first-level sector[§].

Furthermore, one introduces a *first-level Planck constant* $\hbar_{(1)}$ as a new expansion parameter for the quantum action

$$X \;=\; \Omega + (i\hbar_{(1)})\Xi + (i\hbar_{(1)})^2\tilde{\Omega} + \mathcal{O}(\hbar^3_{(1)}) \;, \tag{3.12}$$

where the dependence of the λ^α's, the λ^*_α's and the previous (zeroth) level objects is implied. At the end of the calculations one substitutes back $\hbar_{(1)} \to \hbar$. The (first-level) Planck number grading $\mathsf{Pl} \equiv \mathsf{Pl}_{(1)}$ is defined as [4]

$$\mathsf{Pl}(F \cdot G) \;=\; \mathsf{Pl}(F) + \mathsf{Pl}(G) \;, \quad \mathsf{Pl}(\hbar_{(0)}) \;=\; \mathsf{Pl}(\Gamma^A) \;=\; 0 \;, \quad \mathsf{Pl}(\hbar_{(1)}) \;=\; \mathsf{Pl}(\lambda^\alpha) \;=\; -\mathsf{Pl}(\lambda^*_\alpha) \;=\; 1 \;. \tag{3.13}$$

One may compactly write the Planck number grading as a Planck number operator

$$\mathsf{Pl} \;=\; -(\lambda^\alpha\lambda^*_\alpha, \;\cdot\;)_{[1]} + \hbar_{(1)}\frac{\partial}{\partial\hbar_{(1)}} \;. \tag{3.14}$$

We remark that the introduction of two different expansion parameters $\hbar_{(0)}$ and $\hbar_{(1)}$ is spurred on one hand by the wish to limit the number of terms in X by imposing (first-level) Planck number conservation, and on the other hand to allow Ω, Ξ, $\tilde{\Omega}$, etc., to depend on $\hbar_{(0)}$. If the latter is not an issue, one only needs one Planck constant.

At the first level one is guided by the following

Principle 3.1 *The gauge-fixing action X satisfies three requirements:*

1. *Planck number conservation:* $\mathsf{Pl}(\frac{X}{\hbar_{(1)}}) = 0$.

[‡]Notation: We use capital roman letters A, B, C, ... from the beginning of the alphabet as upper index for both Γ^A and $\Gamma^A_{[1]}$, respectively. Usually a quantity $Q_{(n)}$ with a soft-bracket index (n) is associated with the $n'th$-level only, while a quantity $Q_{[n]}$ with a hard-bracket index $[n]$ accumulates all the levels $\leq n$.

[§]In the multi-level formalism the previous levels are treated covariantly and the present level non-covariantly. This means at the first-level that general zeroth-level coordinate transformations $\Gamma^A \to \Gamma'^A$ are allowed, while the first-level fields λ^α and λ^*_α are considered to be fixed from the onset. This implies for instance that it is consistent to choose a trivial measure in the first-level sector.

2. *The Quantum Master Equation:* $\Delta_{[1]} \exp\left[\frac{i}{\hbar_{(1)}} X\right] = 0.$

3. *The Hessian of X should have rank equal to half the number of fields $\Gamma^A_{[1]}$, i.e. $2N$ in the irreducible case.*

Strickly speaking, it is enough that the rank conditions are met only on stationary field configurations, a technicality we shall assume implicitly from now on. Planck number conservation limits the lowest-order terms in X to

$$X = G_\alpha \lambda^\alpha + (i\hbar_{(1)})H - \lambda^*_\alpha R^\alpha + \mathcal{O}((\lambda^*)^2)\,, \qquad \mathsf{Pl}(X) = 1\,, \qquad (3.15)$$

where

$$-R^\alpha = \frac{1}{2}U^\alpha_{\beta\gamma}\,\lambda^\gamma\lambda^\beta(-1)^{\epsilon_\beta+1} + (i\hbar_{(1)})V^\alpha{}_\beta\lambda^\beta + (i\hbar_{(1)})^2\,\tilde{G}^\alpha\,, \qquad \mathsf{Pl}(R^\alpha) = 2\,. \qquad (3.16)$$

The Quantum Master Equation generates a tower of equations; the first few read

$$(G_\alpha, G_\beta) = G_\gamma U^\gamma_{\alpha\beta}\,, \qquad (3.17)$$

$$(\Delta G_\beta) - (H, G_\beta) = (-1)^{\epsilon_\alpha} U^\alpha_{\alpha\beta} + G_\alpha V^\alpha{}_\beta\,, \qquad (3.18)$$

$$e^H(\Delta e^{-H}) = -(\Delta H) + \frac{1}{2}(H, H) = V^\alpha{}_\alpha - G_\alpha \tilde{G}^\alpha\,. \qquad (3.19)$$

The most important eq. (3.17) is a non-Abelian involution of the N gauge-fixing constraints G_α.

There are essentially two equivalent ways of performing *first-level gauge-fixing*: One may gauge-fix either the X or the W-part.

1a. To gauge-fix the X-part, first change the Boltzmann factor $e^{\frac{i}{\hbar}X}$ with a $\Delta_{[1]}$-exact transformation generated by a gauge fermion Ψ,

$$e^{\frac{i}{\hbar}X^\Psi} = e^{[\overrightarrow{\Delta}_{[1]}, \Psi]}e^{\frac{i}{\hbar}X}\,. \qquad (3.20)$$

1b. Next put the Lagrange multiplier antifields $\lambda^*_\alpha \to 0$ to zero,

$$\mathcal{Z}^\Psi_{[1]} = \int [d\Gamma][d\lambda]\,\rho\,e^{\frac{i}{\hbar}(W+X^\Psi)}\Big|_{\lambda^*=0} = \int [d\Gamma_{[1]}]\,\rho\,\delta(\lambda^*)\,e^{\frac{i}{\hbar}(W+X^\Psi)}\,. \qquad (3.21)$$

1c. The partition function $\mathcal{Z}^\Psi_{[1]}$ defined this way does not depend on the gauge fermion Ψ. PROOF: Use second-level techniques, cf. Section 6, and exponentiate the δ-function in eq. (3.21):

$$\delta(\lambda^*)e^{\frac{i}{\hbar}W} = \int [d\lambda_{(2)}]e^{\frac{i}{\hbar}W_{[2]}} \qquad (3.22)$$

with an action $W_{[2]} = \lambda^*_\alpha \lambda^\alpha_{(2)} + W$ that satisfies the Master Equation $\Delta_{[2]}e^{\frac{i}{\hbar}W_{[2]}} = 0$. Then the partition function

$$\mathcal{Z}^\Psi_{[1]} = \int [d\Gamma_{[1]}][d\lambda_{(2)}]\rho\,e^{\frac{i}{\hbar}W_{[2]}}e^{[\overrightarrow{\Delta}_{[2]}, \Psi]}e^{\frac{i}{\hbar}X} \qquad (3.23)$$

becomes of the W-X–form discussed in the Introduction.

$\square$

1d. If furthermore the gauge fermion Ψ is independent of the Lagrange multiplier antifields λ_α^*, as is normally assumed, the gauge-fixing action (3.20) reduces to

$$X^\Psi = e^{-\mathrm{ad}_{[1]}\Psi}X - (i\hbar)E(-\mathrm{ad}\Psi)\Delta\Psi , \qquad (3.24)$$

where in detail,

$$e^{-\mathrm{ad}_{[1]}\Psi}X = X\left(e^{-\mathrm{ad}\Psi}\Gamma; \lambda, \lambda^* - E(-\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\lambda}\right) . \qquad (3.25)$$

2a. To alternatively gauge-fix the W-part, one may use the symmetry (1.1) of the $\Delta_{[1]}$-operator to re-write eq. (3.21) as

$$\mathcal{Z}_{[1]}^\Psi = \int [d\Gamma_{[1]}] \, \rho \, e^{\frac{i}{\hbar}X} e^{-[\vec{\Delta}_{[1]},\Psi]}\delta(\lambda^*)e^{\frac{i}{\hbar}W} = \int [d\Gamma_{[1]}] \, \rho \, e^{\frac{i}{\hbar}X} \, \delta\left(e^{\mathrm{ad}_{[1]}\Psi}\lambda^*\right) e^{\frac{i}{\hbar}W_{[1]}^\Psi} , \qquad (3.26)$$

where we have defined

$$e^{\frac{i}{\hbar}W_{[1]}^\Psi} = e^{-[\vec{\Delta}_{[1]},\Psi]}e^{\frac{i}{\hbar}W} . \qquad (3.27)$$

2b. If the gauge fermion Ψ is independent of the Lagrange multiplier antifields λ_α^*, the action $W_{[1]}^\Psi$ in eq. (3.27) reduces to W^Ψ in eq. (3.2), and the Lagrange multiplier antifields are gauge-fixed as

$$\lambda_\alpha^* = -E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\lambda^\alpha} , \qquad (3.28)$$

i.e. the partition function reads

$$\mathcal{Z}_{[1]}^\Psi = \int [d\Gamma][d\lambda] \, \rho \, e^{\frac{i}{\hbar}(W^\Psi+X)}\Big|_{\lambda^*=-E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\lambda}} . \qquad (3.29)$$

3.3 Going On-shell with respect to the Constraints

A tractable gauge is the $\lambda_\alpha^* \equiv 0$ gauge, i.e. a trivial gauge fermion $\Psi \equiv 0$. Then X reduces to only two terms

$$X|_{\lambda^*=0} = G_\alpha\lambda^\alpha + (i\hbar_{(1)})H . \qquad (3.30)$$

The λ^α's becomes Lagrange multipliers for the constraints G_α, which are in turn enforced directly through δ-functions in the path integral – hence the name. Moreover, the set of constraints G_α has to be irreducible in order for the rank condition on the Hessian of X to be met, i.e.

$$\forall X^\alpha : G_\alpha X^\alpha = 0 \quad \Rightarrow \quad \exists A^{\alpha\beta} = -(-1)^{\epsilon_\alpha\epsilon_\beta}A^{\beta\alpha} : X^\alpha = G_\beta A^{\beta\alpha} . \qquad (3.31)$$

Let $F^\alpha = F^\alpha(\Gamma;\hbar)$ be arbitrary zeroth-level coordinate functions of statistics $\epsilon(F^\alpha) = \epsilon_\alpha+1$ such that $\bar{\Gamma}^A \equiv \{F^\alpha; G_\alpha\}$ forms a coordinate system in the zeroth-level sector, and let $J = \mathrm{sdet}(\frac{\partial\bar{\Gamma}^A}{\partial\Gamma^B})$ denote the Jacobian of the transformation $\Gamma^A \to \bar{\Gamma}^A$. Then

Theorem 3.2 : *The Quantum Master Equation implies that the quantum-correction H depends on the constraints G_α modulo terms that vanish on-shell with respect to the G_α's according to the following square root formula [5, 13]*

$$H = -\ln\sqrt{\frac{J \, \mathrm{sdet}(F^\alpha, G_\beta)}{\rho}} + \mathcal{O}(G) \qquad (3.32)$$

up to an overall unphysical integration constant that may be discarded. Moreover in the Abelian case $U_{\alpha\beta}^\gamma = 0$, it is possible to solve off-shell as well:

$$H = -\ln\sqrt{\frac{J \, \mathrm{sdet}(F^\alpha, G_\beta)}{\rho}} - G_\alpha \int_0^1 V^\alpha_{\ \beta}\left((F,G)^{-1}\right)^\beta_{\ \gamma}\Big|_{(F;G)\to(tF;G)} dt \, F^\gamma . \qquad (3.33)$$

16

PROOF OF THEOREM 3.2: First note that the $N \times N$ matrix $\Lambda^\alpha{}_\beta \equiv (F^\alpha, G_\beta)$ is invertible, at least in a neighborhood of the constrained surface $G_\alpha \approx 0$. Use the Jacobi identity and the involution (3.17)

$$((F^\gamma, G_\alpha), G_\beta) - (-1)^{(\epsilon_\alpha+1)(\epsilon_\beta+1)}(\alpha \leftrightarrow \beta) = (F^\gamma, (G_\alpha, G_\beta)) = (F^\gamma, G_\delta U^\delta_{\alpha\beta}) \tag{3.34}$$

to deduce that

$$(\Lambda^{-1})^\gamma{}_\delta(\Lambda^\delta{}_\alpha, G_\beta) - (-1)^{(\epsilon_\alpha+1)(\epsilon_\beta+1)}(\alpha \leftrightarrow \beta) = U^\gamma_{\alpha\beta} + (-1)^{\epsilon_\gamma\epsilon_\delta}G_\delta(\Lambda^{-1})^\gamma{}_\epsilon(F^\epsilon, U^\delta_{\alpha\beta}) . \tag{3.35}$$

Now supertrace to get

$$(\operatorname{str}\ln\Lambda, G_\beta) + (-1)^{(\epsilon_\alpha+1)\epsilon_\beta}(\Lambda^\alpha{}_\beta, G_\gamma)(\Lambda^{-1})^\gamma{}_\alpha = (-1)^{\epsilon_\alpha}U^\alpha_{\alpha\beta} + (-1)^{\epsilon_\alpha(\epsilon_\gamma+1)}G_\gamma(\Lambda^{-1})^\alpha{}_\delta(F^\delta, U^\gamma_{\alpha\beta}) . \tag{3.36}$$

Next use the new coordinates $\bar{\Gamma}^A \equiv \{F^\alpha; G_\alpha\}$ to rewrite

$$
\begin{aligned}
(\Delta G_\beta) &= \frac{(-1)^{\epsilon_A}}{2\bar\rho} \frac{\overrightarrow{\partial^l}}{\partial\bar\Gamma^A}\bar\rho(\bar\Gamma^A, G_\beta) \\
&= (\ln\sqrt{\bar\rho}, G_\beta) + \frac{(-1)^{\epsilon_\alpha+1}}{2}\frac{\overrightarrow{\partial^l}}{\partial F^\alpha}(F^\alpha, G_\beta) + \frac{(-1)^{\epsilon_\alpha}}{2}\frac{\overrightarrow{\partial^l}}{\partial G_\alpha}(G_\alpha, G_\beta) \\
&= (\ln\sqrt{\bar\rho}, G_\beta) + \frac{(-1)^{(\epsilon_\alpha+1)\epsilon_\beta}}{2}(\Lambda^\alpha{}_\beta\frac{\overleftarrow{\partial^r}}{\partial F^\alpha}) + \frac{(-1)^{\epsilon_\alpha}}{2}\frac{\overrightarrow{\partial^l}}{\partial G_\alpha}[G_\gamma U^\gamma_{\alpha\beta}] \\
&= (\ln\sqrt{\frac{\bar\rho}{\operatorname{sdet}\Lambda}}, G_\beta) + (-1)^{\epsilon_\alpha}U^\alpha_{\alpha\beta} + (-1)^{\epsilon_\alpha(\epsilon_\gamma+1)}G_\gamma\frac{\overrightarrow{\partial^l}}{\partial G_\alpha}U^\gamma_{\alpha\beta} \\
&\quad - \frac{(-1)^{(\epsilon_\alpha+1)\epsilon_\beta}}{2}(\Lambda^\alpha{}_\beta\frac{\overleftarrow{\partial^r}}{\partial G_\gamma})(G_\gamma, G_\delta)(\Lambda^{-1})^\delta{}_\alpha \\
&\quad + \frac{(-1)^{\epsilon_\alpha(\epsilon_\gamma+1)}}{2}G_\gamma(\Lambda^{-1})^\alpha{}_\delta(F^\delta, F^\epsilon)\frac{\overrightarrow{\partial^l}}{\partial F^\epsilon}U^\gamma_{\alpha\beta} ,
\end{aligned}
\tag{3.37}
$$

where $\bar\rho = \frac{\rho}{J}$ denotes the transformed density, and where eq. (3.36) has been used in the last equality. The last three terms are of order $\mathcal{O}(G)$. Then one of the consequences of the Master Equation (3.18) is

$$\left[\ln\sqrt{\frac{\operatorname{sdet}\Lambda}{\bar\rho}} + H\right]\frac{\overleftarrow{\partial^r}}{\partial F^\beta} = \mathcal{O}(G) , \tag{3.38}$$

and the square root formula (3.32) follows by integration. The integration constant represents the trivial dilation symmetry $H \to H + c$, where c is a constant. Clearly H only appears in differentiated form in the Quantum Master Equation, so any integration constant is a priori allowed. Restricting ourselves to consider only classes of solutions that are $\Delta_{[1]}$-exactly connected, it is consistent to always set this integration constant to zero.

$\square$

The square root formula (3.32) gives H as a function of the F^α's. But since the F^α's are arbitrary, this dependence must be trivial. This fact may also be shown directly:

Lemma 3.3 [5, 13]: The factor $J\operatorname{sdet}(F^\alpha, G_\beta)$ is independent of the F^α's up to terms that vanish on-shell with respect to G_α, if the G_α's satisfy the involution eq. (3.17).

PROOF OF LEMMA 3.3: Exponentiate the two determinants by introducing a ghost pair $\bar{C}_A$ and C^A of statistics ϵ_A+1, and another ghost pair $\bar{B}_\alpha$ and B^α of statistics $\epsilon_\alpha+1$, so that the product of the two determinants inside the square root can be written as a partition function

$$\mathcal{Z}_{\text{det}} \;=\; J\,\text{sdet}(F^\alpha, G_\beta) \;=\; \int [d\bar{C}][dC][d\bar{B}][dB]\; e^{\frac{i}{\hbar}S_{\text{det}}} \tag{3.39}$$

with a determinant action given as

$$S_{\text{det}} \;=\; \bar{C}_A(\bar{\Gamma}^A\frac{\overleftarrow{\partial^r}}{\partial\Gamma^B})C^B + \bar{B}_\alpha(F^\alpha, G_\beta)B^\beta \;. \tag{3.40}$$

Now there are several ways to proceed. Perhaps the most enlightening treatment is to rewrite this as a mini field-antifield system within our theory. Consider a "classical" action

$$S_0 \;=\; \bar{C}^\alpha(G_\alpha\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A \;, \tag{3.41}$$

where we have split $\bar{C}_A = \{\bar{C}_\alpha; \bar{C}^\alpha\}$ of Grassmann parity $\epsilon(\bar{C}_\alpha) = \epsilon_\alpha$ and $\epsilon(\bar{C}^\alpha) = \epsilon_\alpha+1$, respectively. On-shell with respect to G_α the classical action S_0 is invariant $\delta S_0 \approx 0$ under the following BRST-like symmetry

$$\delta C^A \;=\; (\Gamma^A, G_\alpha)B^\alpha\mu \;, \tag{3.42}$$

because of the involution (3.17). Here μ is a Grassmann-odd parameter. The standard field-antifield recipe [1, 2] now instructs us to construct a minimal proper action as

$$S_{\text{min}} \;=\; S_0 + C_A^*(\Gamma^A, G_\alpha)B^\alpha \;, \tag{3.43}$$

and a non-minimal proper action

$$S \;=\; S_{\text{min}} + \bar{C}_\alpha\bar{B}^{*\alpha} \;, \tag{3.44}$$

where we have identified $\bar{C}_\alpha$ with Nakanishi-Lautrup auxiliary fields. It is easy to check that $(S, S)_{BC} \approx 0$ and $\Delta_{BC}S = 0$, where

$$\Delta_{BC} \;\equiv\; (-1)^{\epsilon_\alpha+1}\frac{\overrightarrow{\partial^l}}{\partial B^\alpha}\frac{\overrightarrow{\partial^l}}{\partial B_\alpha^*} + (-1)^{\epsilon_\alpha+1}\frac{\overrightarrow{\partial^l}}{\partial\bar{B}_\alpha}\frac{\overrightarrow{\partial^l}}{\partial\bar{B}^{*\alpha}} + (-1)^{\epsilon_A+1}\frac{\overrightarrow{\partial^l}}{\partial C^A}\frac{\overrightarrow{\partial^l}}{\partial C_A^*} + (-1)^{\epsilon_A+1}\frac{\overrightarrow{\partial^l}}{\partial\bar{C}_A}\frac{\overrightarrow{\partial^l}}{\partial\bar{C}^{*A}} \;. \tag{3.45}$$

So the Quantum Master Equation for S is satisfied on-shell. Now choose a gauge fermion as

$$\Psi \;=\; \bar{B}_\alpha(F^\alpha\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A \;. \tag{3.46}$$

The gauge-fixed action

$$S\left(C_A^* = \frac{\partial\Psi}{\partial C^A}; \bar{B}^{*\alpha} = \frac{\partial\Psi}{\partial\bar{B}_\alpha}\right) \;=\; \bar{C}^\alpha(G_\alpha\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A + \bar{C}_\alpha(F^\alpha\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A + \bar{B}_\alpha(F^\alpha, G_\beta)B^\beta \;=\; S_{\text{det}} \tag{3.47}$$

is precisely the determinant action S_{det}. Hence the partition function (3.39) does not depend on Ψ, and since the F^α's only appear inside the gauge fermion Ψ, we conclude that the partition function (3.39) does not depend on the F^α's as well.

$\square$

The following Theorem 3.4 is in some respect a reversed statement of Theorem 3.2:

Theorem 3.4 : *For an arbitrary set of irreducible constraints G_α and structure functions $U^\gamma_{\alpha\beta}$ that satisfy the involution (3.17), there exist functions H, $V^\alpha{}_\beta$, $\tilde{G}^\alpha$, etc., such that X is a solution to the Quantum Master Equation.*

PROOF: This relies on Abelianization, *i.e.* there exist Abelian constraints G^0_α with $(G^0_\alpha, G^0_\beta) = 0$, and an invertible rotation matrix $\Lambda^\alpha{}_\beta$, such that $G_\beta = G^0_\alpha \Lambda^\alpha{}_\beta$.

$\square$

Corollary 3.5 *[5, 13]: The partition function*[¶]

$$ \mathcal{Z}^G_{[1]} \;=\; \int [d\Gamma]\; e^{\frac{i}{\hbar}W} \delta(G) \sqrt{\rho\; J\; \mathrm{sdet}(F^\alpha, G_\beta)} \tag{3.49} $$

is independent of the G_α's satisfying the involution eq. (3.17).

SKETCHED PROOF: The Corollary does not explicitly refer to an X-part, but we may always assume an underlying X-part because of Theorem 3.4. Therefore the broad strategies concerning independence of the gauge-fixing X-part mentioned in the Introduction apply.

$\square$

Another interesting result is the following

Theorem 3.6 : *The on-shell square root formula eq. (3.32) for H, viewed as part of X, is form invariant under finite $\Delta_{[1]}$-exact deformations of X, even if X does not solve the Quantum Master Equation.*

PROOF OF THEOREM 3.6: Assume that the gauge-fixing action X_f that contains the investigated quantum correction H_f, is a $\Delta_{[1]}$-exact deformation

$$ X_f \;=\; e^{\mathrm{ad}_{[1]}\Psi}X_i + (i\hbar_{(1)})E(\mathrm{ad}_{[1]}\Psi)\Delta_{[1]}\Psi \;, \tag{3.50} $$

of an initial gauge-fixing action X_i with a one-loop correction H_i that obeys the on-shell square root formula eq. (3.32), *i.e.*

$$ H_i \;=\; c - \ln \sqrt{\mathrm{sdet}(\frac{\partial\{F_i; G_i\}}{\partial\Gamma}) \frac{\mathrm{sdet}(F_i^\alpha, G_{i,\beta})}{\rho}} + \mathcal{O}(G_i) \;, \tag{3.51} $$

where we included the integration constant c. We have to show that a similar formula holds for H_f. Order by order in $\hbar_{(1)}$ the eq. (3.50) implies that

$$ \Omega_f \;=\; e^{\mathrm{ad}_{[1]}\Psi_0}\Omega_i \;, \tag{3.52} $$

$$ \Xi_f \;=\; e^{\mathrm{ad}_{[1]}\Psi_0}\Xi_i + E(\mathrm{ad}_{[1]}\Psi_0)\Delta_{[1]}\Psi_0 + \Big(E(\mathrm{ad}_{[1]}\Psi_0)\Psi_1,\; \Omega_f\Big)_{[1]} \;, \tag{3.53} $$

$$ \tilde{\Omega}_f \;=\; \mathcal{O}(\lambda^*) \;. \tag{3.54} $$

[¶]To recover the zeroth-level gauge-fixing (3.9) one first goes to Darboux coordinates $\Gamma^A = \{\phi^\alpha; \phi^*_\alpha\}$ with $\rho = 1$, and then substitute $F^\alpha \to \phi^\alpha$ and $G_\alpha \to \phi^*_\alpha - \partial\Psi/\partial\phi^\alpha$. In our conventions the Lagrange multipliers λ^α and the antifields ϕ^*_α (of the previous level) carry the *same* Grassmann parity. In detail, we define

$$ \epsilon(\phi^\alpha) \;\equiv\; \epsilon^{(0)}_\alpha \;, \qquad \epsilon(\phi^*_\alpha) \;\equiv\; \epsilon^{(0)}_\alpha + 1 \;\equiv\; \epsilon^{(1)}_\alpha \;, \qquad \epsilon(\lambda^\alpha) \;\equiv\; \epsilon^{(1)}_\alpha \;\equiv\; \epsilon_\alpha \;, \tag{3.48} $$

and so forth.

The generator $\Psi = \Psi_0 + (i\hbar_{(1)})\Psi_1 + \mathcal{O}(\hbar_{(1)}^2)$ conserves the Planck number, $\mathsf{Pl}(\Psi) = 0$. This restricts the possible lowest terms to

$$\Psi_0 = \Psi_{(0)} + (-1)^{\epsilon_\alpha + \epsilon_\beta} \lambda^\alpha \Psi_{(1)\alpha}{}^\beta \lambda_\beta^* + \mathcal{O}((\lambda^*)^2) , \tag{3.55}$$

$$\Psi_1 = \mathcal{O}(\lambda^*) , \tag{3.56}$$

where the sign factors in front of the matrix $\Psi_{(1)\alpha}{}^\beta$ of Grassmann grading $\epsilon_\alpha + \epsilon_\beta$ are introduced for later convenience. First we look at the eq. (3.52) for Ω_f:

$$G_{f,\alpha} \lambda^\alpha = \Omega_f|_{\lambda^*=0} = \exp\left[\mathrm{ad}\Psi_{(0)} - (-1)^{\epsilon_\alpha + \epsilon_\beta} \lambda^\alpha \Psi_{(1)\alpha}{}^\beta \frac{\overrightarrow{\partial^l}}{\partial \lambda^\beta} \right] G_{i,\gamma} \lambda^\gamma . \tag{3.57}$$

The constraints $G_{f,\alpha}$ are a composition of a rotation and an anticanonical transformation,

$$G_{f,\alpha} = \Lambda_\alpha{}^\beta \tilde{G}_{i,\beta} , \tag{3.58}$$

where tilde $\sim$ denotes the anticanonical transformation

$$\Gamma^A \longrightarrow \tilde{\Gamma}^A \equiv \exp\left[\mathrm{ad}\Psi_{(0)} \right] \Gamma^A . \tag{3.59}$$

We shall later need an expression for the superdeterminant of the rotation matrix $\Lambda_\alpha{}^\beta$,

$$\ln \mathrm{sdet}\Lambda = \mathrm{str}\ln \Lambda = -E(\mathrm{ad}\Psi_{(0)})\mathrm{str}\Psi_{(1)} . \tag{3.60}$$

To prove the eqs. (3.58) and (3.60) one may use one-parameter techniques, i.e. let the generator $\Psi \to t\Psi$ be proportional to a parameter $t \in [0,1]$ to study the transition

$$G_\alpha(t{=}0) \equiv G_{i,\alpha} \longrightarrow G_\alpha(t{=}1) \equiv G_{f,\alpha} . \tag{3.61}$$

It follows from eq. (3.57) that the constraints obey the differential equation

$$\frac{dG_\alpha(t)}{dt} = (\Psi_{(0)}, G_\alpha(t)) - \Psi_{(1)\alpha}{}^\beta G_\beta(t) . \tag{3.62}$$

The first term on the right hand side represents an anticanonical transformation, while the second term is a rotation. The solution to eq. (3.62) is

$$G_\alpha(t) = \Lambda_\alpha{}^\beta(t) \exp\left[t\, \mathrm{ad}\Psi_{(0)} \right] G_{i,\beta} , \tag{3.63}$$

where the rotation matrix $\Lambda(t)$ is a path-ordered matrix expression in the parameter $t' \in [0,t]$,

$$\Lambda(t) = \mathcal{P} \exp\left[-\int_0^t dt'\, e^{(t-t')\mathrm{ad}\Psi_{(0)}} \Psi_{(1)} \right] . \tag{3.64}$$

This leads immediately to eqs. (3.58) and (3.60).

Next we look at the right hand side of the eq. (3.53) for Ξ_f. The third term is proportional to either the constraints $G_{f,\alpha}$ or to λ_α^* because of eqs. (3.56) and (3.57), so only the first two terms contribute on-shell:

$$H_f = \Xi_f|_{\lambda^*=0} = \exp\left[\mathrm{ad}\Psi_{(0)} \right] H_i + E(\mathrm{ad}\Psi_{(0)})\left(\Delta\Psi_{(0)} + \mathrm{str}\Psi_{(1)} \right) + \mathcal{O}(G_f) . \tag{3.65}$$

Combining eq. (3.65), (3.51), (2.60), (3.60) and (3.58), one deduces the Theorem:

$$-H_f = -\tilde{c} + \ln \sqrt{ \mathrm{sdet}(\frac{\partial\{\tilde{F}_i; \tilde{G}_i\}}{\partial\tilde{\Gamma}}) \frac{\mathrm{sdet}(\tilde{F}_i^\alpha, \tilde{G}_{i,\beta})}{\rho(\tilde{\Gamma})} } + \mathcal{O}(\tilde{G}_i)$$

$$+ \ln \sqrt{\frac{\rho(\tilde{\Gamma})}{\rho} \, \mathrm{sdet}(\frac{\partial \tilde{\Gamma}^A}{\partial \Gamma^B})} + \ln \mathrm{sdet}\Lambda + \mathcal{O}(G_f)$$

$$= -c + \ln \sqrt{\mathrm{sdet}(\frac{\partial \{\tilde{F}_i; \tilde{G}_i\}}{\partial \Gamma}) \, \mathrm{sdet}\Lambda^2 \frac{\mathrm{sdet}(\tilde{F}_i^\alpha, \tilde{G}_{i,\beta})}{\rho}} + \mathcal{O}(G_f)$$

$$= -c + \ln \sqrt{\mathrm{sdet}(\frac{\partial \{F_f; G_f\}}{\partial \Gamma}) \, \frac{\mathrm{sdet}(F_f^\alpha, G_{f,\beta})}{\rho}} + \mathcal{O}(G_f) \, , \tag{3.66}$$

with $F_f^\alpha = \tilde{F}_i^\alpha$.

$\square$

Since one may in principle create every X-solution through $\Delta_{[1]}$-exact deformations of some trivial X-action like that of the $\phi_\alpha^* = 0$ gauge, one may interpret Theorem 3.6 as generating the square root formula (3.32) via first-level anticanonical transformations $e^{\mathrm{ad}\Psi}$. Theorem 3.6 also shows that the integration constant from Theorem 3.2 is invariant under $\Delta_{[1]}$-exact deformations, so one may consistently discard it. The proof shows that only the classical part Ψ_0 of the underlying generator Ψ plays an active rôle in the transformation of H on-shell. Here the word *classical* is used in the first-level sense, *i.e.* for objects independent of $\hbar_{(1)}$. Moreover, we have seen that Ψ_0 generates rotations and zeroth-level anticanonical transformations of the G_α's.

4 Second-Class Constraints

It is of interest to extend the irreducible first-level construction of a gauge-fixed Lagrangian path integral to include antisymplectic second-class constraints [4, 14]. Consider therefore a set of $2N_D$ second-class constraints Θ^a with Grassmann parity $\epsilon(\Theta^a) = \epsilon_a$ that reduce the $2N$-dimensional antisymplectic manifold down to a physical submanifold of dimension $2(N - N_D)$. This proceeds quite analogous to the Poisson-bracket treatment of second-class constraints in the Hamiltonian formalism[||]. The antibracket matrix

$$E^{ab} \equiv (\Theta^a, \Theta^b) \tag{4.1}$$

of the second-class constraints Θ^a has by definition an inverse matrix E_{ab},

$$E_{ab}E^{bc} = \delta_a^c \, , \tag{4.2}$$

so that one can introduce a Dirac antibracket completely analogous to the Dirac Poisson bracket [4]

$$(F, G)_D \equiv (F, G) - (F, \Theta^a) E_{ab} (\Theta^b, G) \, , \tag{4.3}$$

where $F = F(\Gamma)$ and $G = G(\Gamma)$ are arbitrary functions. The bracket satisfies a Jacobi identity

$$\sum_{F,G,H \text{ cycl.}} (-1)^{(\epsilon_F + 1)(\epsilon_H + 1)} ((F, G)_D, H)_D = 0 \tag{4.4}$$

[||]Here, we work partly at the "gauge-generating" zeroth-level and partly at the "gauge-fixing" first-level. Therefore the second-class constraints $\Theta^a = \Theta^a(\Gamma; \hbar_{(-1)})$ and several of the Dirac constructions to be introduced below could in principle depend on a Planck expansion parameter $\hbar_{(-1)}$, which we assign to a previous "minus first" level. Moreover, we postpone for simplicity the issue of reparametrizations of the constraints $\Theta^a \to \Theta'^a = \Lambda^a{}_b(\Gamma) \, \Theta^b$ to Section 4.5, *i.e.* the *defining* set of constraints are kept fixed for now. In fact we derive in Section 4.5 that a reparametrization invariant formulation *necessitates* off-shell corrections to eqs. (4.5), (4.6) and (4.11). Finally, let us mention that antisymplectic *first-class* constraints, and moreover, the conversion from second to first-class antisymplectic constraints, have been addressed in [15].

everywhere in the extended phase space Γ^A. The projection property ensures that the Dirac antibracket vanishes,

$$(\mathrm{ad}_D \Theta^a) F \equiv (\Theta^a, F)_D = 0 , \tag{4.5}$$

when taken of any function F with any of the constraints Θ^a. In addition, there exists a nilpotent Dirac Δ-operator Δ_D,

$$\Delta_D^2 = 0 , \tag{4.6}$$

so that the Dirac antibracket (4.3) equals the failure of Δ_D to act as a derivation, in complete analogy with the usual Δ-operator. It reads

$$\Delta_D = \frac{(-1)^{\epsilon_A}}{2\rho_D} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} \rho_D E_D^{AB} \frac{\overrightarrow{\partial^l}}{\partial \Gamma^B} , \tag{4.7}$$

with a degenerated antisymplectic metric

$$E_D^{AB} \equiv (\Gamma^A, \Gamma^B)_D , \tag{4.8}$$

and with a compatible Dirac measure density $\rho_D = \rho_D(\Gamma)$. By definition the Dirac measure density ρ_D transforms as

$$\rho_D' = \frac{\rho_D}{\mathrm{sdet}(\frac{\partial \Gamma'^A}{\partial \Gamma^B})} \tag{4.9}$$

under change of coordinates $\Gamma^A \rightarrow \Gamma'^A$, while the Dirac measure density ρ_D is required to transform as

$$\rho_D' = \rho_D \, \mathrm{sdet}(\Lambda^a{}_b) + \mathcal{O}(\Theta) \tag{4.10}$$

under reparametrization of the constraints $\Theta^a \rightarrow \Theta'^a = \Lambda^a{}_b(\Gamma) \, \Theta^b$. Finally, the Δ_D-operator annihilates the constraints

$$(\Delta_D \Theta^a) = 0 , \tag{4.11}$$

because of eq. (4.5), and independently of the choice of ρ_D. In the case of higher-order Δ-operators the Θ^a's become *operators*, and the condition (4.11) should be replaced with $[\Delta_D, \Theta^a] = 0$, cf. [14].

4.1 First-Level Partition Function

With the above ingredients, the corresponding irreducible first-level Lagrangian path integral formulation can be carried out very analogous to the case without second-class constraints [4]. The appropriate path integral in the $\lambda_\alpha^* = 0$ gauge, is,

$$\mathcal{Z}_{[1]D} = \int [d\Gamma][d\lambda] \, \rho_D \, e^{\frac{i}{\hbar}(W_D + X_D)} \prod_a \delta(\Theta^a) , \tag{4.12}$$

with both W_D and X_D satisfying the corresponding Quantum Master Equations

$$\Delta_D \exp \left[\frac{i}{\hbar_{(0)}} W_D \right] = 0 , \qquad\qquad \Delta_{[1]D} \exp \left[\frac{i}{\hbar_{(1)}} X_D \right] = 0 . \tag{4.13}$$

At the first-level there are $N - N_D$ Lagrange multipliers λ^α and $N - N_D$ corresponding antifields λ_α^*. One may again expand the action

$$X_D = G_\alpha \lambda^\alpha + (i\hbar_{(1)}) H + \mathcal{O}(\lambda^*) \tag{4.14}$$

in terms allowed by the Planck number conservation. The Quantum Master Equation for X_D shows that the $N - N_D$ gauge-fixing functions G_α are in involution with respect to the Dirac antibracket,

$$(G_\alpha, G_\beta)_D = G_\gamma U_{\alpha\beta}^\gamma . \tag{4.15}$$

22

As in the case with no second-class constraints, an on-shell closed-form expression for the one-loop correction H has been found [14]. Let $F^\alpha = F^\alpha(\Gamma; \hbar)$ be arbitrary zeroth-level coordinate functions such that

$$\bar{\Gamma}^A \equiv \{F^\alpha; G_\alpha; \Theta^a\} \tag{4.16}$$

forms a coordinate system in the zeroth-level sector, and let $J_D = \text{sdet}(\frac{\partial \bar{\Gamma}^A}{\partial \Gamma^B})$ denote the Jacobian of the transformation $\Gamma^A \to \bar{\Gamma}^A$. Then

Theorem 4.1 : *The Quantum Master Equation implies that the one-loop correction H depends on the constraints G_α modulo terms that vanish on-shell with respect to the G_α's and the Θ^a's according to the following square root formula*

$$H \;=\; -\ln \sqrt{\frac{J_D \; \text{sdet}(F^\alpha, G_\beta)_D}{\rho_D}} + \mathcal{O}(G; \Theta) \;. \tag{4.17}$$

One may check that

Lemma 4.2 : *The factor $J_D \, \text{sdet}(F^\alpha, G_\beta)_D$ is independent of the F^α's up to terms that vanish on-shell with respect to G_α, if the G_α's satisfy the involution (4.15) with respect to the Dirac antibracket.*

PROOF OF LEMMA 4.2: We may also this time build an auxiliary field-antifield system. It is almost identical to the case without second-class constraints, so we shall only point out some of the differences. The "classical" action S_0 now reads

$$S_0 \;=\; \bar{C}^\alpha (G_\alpha \frac{\overleftarrow{\partial^r}}{\partial \Gamma^A}) C^A + \bar{C}_a (\Theta^a \frac{\overleftarrow{\partial^r}}{\partial \Gamma^A}) C^A \;, \tag{4.18}$$

where we have split the antighost $\bar{C}_A = \{\bar{C}_\alpha; \bar{C}^\alpha; \bar{C}_a\}$ in three parts that reflects the splitting in (4.16). On-shell with respect to G_α the classical action S_0 is invariant $\delta S_0 \approx 0$ under the following BRST-like symmetry

$$\delta C^A \;=\; (\Gamma^A, G_\alpha)_D B^\alpha \mu \;, \tag{4.19}$$

because of the involution (4.15). Note that in the Lemma we can work off-shell with respect to the Θ^a's. Hence the minimal proper action of this auxiliary field-antifield system is

$$S_{\min} \;=\; S_0 + C_A^* (\Gamma^A, G_\alpha)_D B^\alpha \;, \tag{4.20}$$

and the non-minimal proper action is

$$S \;=\; S_{\min} + \bar{C}_\alpha \bar{B}_*^\alpha \;. \tag{4.21}$$

□

There are at least three very good reasons to impose the ρ_D transformation rule (4.10). First of all, it is precisely what is needed to make the partition function (4.12) invariant under reparametrization of the constraints $\Theta^a \to \Theta'^a = \Lambda^a{}_b(\Gamma) \, \Theta^b$. Secondly, note that the rule (4.10) also render the expression inside the square root of (4.17) reparametrization invariant, up to terms that vanish on-shell with respect to the Θ^a's. Thirdly, we shall show in Section 4.5 below that the rule (4.10) is needed to make the Dirac odd Laplacian Δ_D reparametrization invariant on-shell.

4.2 Unitarizing Coordinates

The $2N$ zeroth-level variables $\Gamma^A = \Gamma^A(\gamma; \Theta)$ can be viewed as functions of $2(N - N_D)$ physical variables γ^A and $2N_D$ second-class variables Θ^a such that the second-class constraints satisfy $\Theta^a(\Gamma(\gamma; \Theta)) = \Theta^a$. In other words, we may choose so-called *unitarizing* coordinates Γ^A that split $\Gamma^A = \{\gamma^A; \Theta^a\}$ directly into a physical and a second-class subsector. We use capital roman letters A, B, C, ... from the beginning of the alphabet as upper index for both the full and the reduced variables Γ^A and γ^A, respectively. A change of unitarizing coordinates

$$\Gamma^A \;=\; \{\gamma^A; \Theta^a\} \quad \longrightarrow \quad \Gamma'^A \;=\; \{\gamma'^A; \Theta'^a\} \tag{4.22}$$

has in general the form

$$\gamma'^A \;=\; \gamma'^A(\Gamma) \;, \qquad\qquad \Theta'^a = \Lambda^a{}_b(\Gamma) \, \Theta^b \;, \tag{4.23}$$

where the matrix $\Lambda^a{}_b$ is invertible. This causes the Jacobian J of the coordinate transformation to factorize on-shell,

$$J \;=\; \mathrm{sdet}\left(\frac{\partial \Gamma'^A}{\partial \Gamma^B}\right) \;=\; \mathrm{sdet}\left(\frac{\partial \gamma'^A}{\partial \gamma^B}\right) \mathrm{sdet}\left(\frac{\partial \Theta'^a}{\partial \Theta^b}\right) + \mathcal{O}(\Theta) \;. \tag{4.24}$$

The Dirac antibracket becomes

$$(F, G)_D \;=\; (F \frac{\overleftarrow{\partial^r}}{\partial \gamma^A})(\gamma^A, \gamma^B)_D (\frac{\overrightarrow{\partial^l}}{\partial \gamma^B} G) \tag{4.25}$$

in unitarizing coordinates.

4.3 Reduction to Physical Submanifold

In unitarizing coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$, we may assign reduced "tilde" objects that live on the physical submanifold. In order of appearance,

$$\begin{aligned}
\tilde{E}^{AB} &\equiv \left. (\gamma^A, \gamma^B)_D \right|_{\Theta=0} \;, \\
\tilde{\rho} &\equiv \left. \rho_D \right|_{\Theta=0} \;, \\
\tilde{\Delta} &\equiv \frac{(-1)^{\epsilon_A}}{2\tilde{\rho}} \frac{\overrightarrow{\partial^l}}{\partial \gamma^A} \tilde{\rho} \tilde{E}^{AB} \frac{\overrightarrow{\partial^l}}{\partial \gamma^B} \;=\; \left. \Delta_D \right|_{\Theta=0} \;, \\
\tilde{W} &\equiv \left. W_D \right|_{\Theta=0} \;, \\
\tilde{X} &\equiv \left. X_D \right|_{\Theta=0} \;, \\
\tilde{G}_\alpha &\equiv \left. G_\alpha \right|_{\Theta=0} \;, \\
\tilde{H} &\equiv \left. H \right|_{\Theta=0} \;, \\
\tilde{F}^\alpha &\equiv \left. F^\alpha \right|_{\Theta=0} \;, \qquad \text{etc.}
\end{aligned} \tag{4.26}$$

The antibracket

$$(F, G)_\sim \;\equiv\; (F \frac{\overleftarrow{\partial^r}}{\partial \gamma^A}) \tilde{E}^{AB} (\frac{\overrightarrow{\partial^l}}{\partial \gamma^B} G) \;, \tag{4.27}$$

the measure $\tilde{\rho}[d\gamma]$, the odd Laplacian $\tilde{\Delta}$, the actions $\tilde{W}$ and $\tilde{X}$, etc., are all independent of the defining set of constraints Θ^a (and of the unitarizing coordinates) used in the Dirac construction, cf. Section 4.5. The odd Laplacian $\tilde{\Delta}$ is nilpotent, $\tilde{\Delta}^2 = 0$, because Δ_D does not contain Θ-derivatives (when using unitarizing coordinates). Furthermore, the actions $\tilde{W}$ and $\tilde{X}$ satisfy the Quantum Master Equations,

$$\tilde{\Delta} \exp\left[\frac{i}{\hbar_{(0)}} \tilde{W}\right] \;=\; 0 \;, \qquad\qquad \tilde{\Delta}_{[1]} \exp\left[\frac{i}{\hbar_{(1)}} \tilde{X}\right] \;=\; 0 \;. \tag{4.28}$$

The first-level partition function (4.12) reduces to

$$\mathcal{Z}_{[1]D} \;\; = \;\; \int [d\gamma][d\lambda] \; \tilde{\rho} \; e^{\frac{i}{\hbar}(\tilde{W}+\tilde{X})} \tag{4.29}$$

on the physical submanifold. The resulting partition function is precisely of the general form (1.3) for fields entirely living on the physical submanifold. Moreover, the coordinates $\bar{\Gamma}^A \equiv \{F^\alpha; G_\alpha; \Theta^a\}$ used in (4.16), and in particular the coordinates $\{\tilde{F}^\alpha; \tilde{G}_\alpha; \Theta^a\}$ are both examples of unitarizing coordinates. Therefore the square root formula (4.17) reduces to

$$\tilde{H} \;\; = \;\; -\ln\sqrt{\frac{\tilde{J}\;\mathrm{sdet}(\tilde{F}^\alpha, \tilde{G}_\beta)_{\sim}}{\tilde{\rho}}} + \mathcal{O}(\tilde{G}) \; . \tag{4.30}$$

Here we have used that the Jacobian

$$J_D|_{\Theta=0} \;\; = \;\; \mathrm{sdet}(\frac{\partial \bar{\Gamma}^A}{\partial \Gamma^B})\Big|_{\Theta=0} \;\; = \;\; \mathrm{sdet}(\frac{\partial\{\tilde{F}^\alpha; \tilde{G}_\alpha\}}{\partial \gamma^B}) \;\; \equiv \;\; \tilde{J} \tag{4.31}$$

satisfies the factorization property (4.24). We summarize the above observations in the following

Theorem 4.3 – Reduction Theorem: *A first-level field-antifield theory (4.12) with second-class constraints Θ^a may always be written in a set of unitarizing coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$. In these coordinates the theory reduces to a physical theory (4.29) with physical coordinates γ^A on the physical submanifold. The reduction is independent of the parametrization of the constraints Θ^a and the choice of unitarizing coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$.*

4.4 Transversal Coordinates

Let us define *transversal* coordinates as unitarizing coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$ with the additional property that

$$(\gamma^A, \Theta^a) \;\; = \;\; 0 \; , \tag{4.32}$$

so that the second-class variables Θ^a and the physical variables γ^A are perpendicular to each other in the antibracket sense. For every system of unitarizing coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$ there exist unique deformation functions $X_a^A = X_a^A(\Gamma)$ such that a unique "primed" set of coordinates $\Gamma'^A = \{\gamma'^A; \Theta'^a\}$, defined as

$$\begin{cases} \gamma'^A \;\; = \;\; \gamma^A - X_a^A \Theta^a \; , \\ \Theta'^a \;\; = \;\; \Theta^a \; , \end{cases} \tag{4.33}$$

is a set of transversal coordinates: $(\gamma'^A, \Theta'^a)=0$. In fact, X_a^A satisfy the following fixed-point equation,

$$X_a^A \;\; = \;\; (\gamma^A, \Theta^b)E_{ba} - (-1)^{\epsilon_c(1+\epsilon_A)}\Theta^c(X_c^A, \Theta^b)E_{ba} \tag{4.34}$$

that may be solved recursively $X_a^A = (\gamma^A, \Theta^b)E_{ba} + \mathcal{O}(\Theta)$ to all orders in Θ.

We conclude that each set of second-class constraints Θ^a may be complemented with variables γ^A into a system of transversal coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$. In transversal coordinates the Dirac antibracket matrix

$$(\gamma^A, \gamma^B)_D \;\; = \;\; (\gamma^A, \gamma^B) \tag{4.35}$$

becomes the original antibracket matrix. This may be used to give a short proof of the remarkable fact that the Jacobi identity (4.4) for the Dirac antibracket holds everywhere in the extended phase space Γ^A. Clearly, for all physical purposes it would have been enough to have the Jacobi identity

(4.4) satisfied just on the physical submanifold. Nevertheless, the Dirac construction (4.3) provides the stronger Jacobi identity (4.4) for free.

Similarly, we may always impose strong nilpotency (4.6) of Δ_D when considering an arbitrary but fixed set of second-class constraints Θ^a. However, we shall see in the next Section 4.5 that strong nilpotency (4.6) and the transformation rule (4.10) cannot both be maintained under reparametrization of the second-class constraints Θ^a. We have already seen the necessity of the transformation rule (4.10), so instead we would surprisingly have to relax the nilpotency requirement (4.6) for Δ_D. A manifestly reparametrization invariant Ansatz turns out to be that the square of the Dirac odd Laplacian,

$$\Delta_D^2 = O^A \frac{\overrightarrow{\partial^l}}{\partial \Gamma^A} , \tag{4.36}$$

is a first order differential operator with coefficient functions $O^A = \mathcal{O}(\Theta)$ that vanish on-shell with respect to the second-class constraints Θ^a.

4.5 Reparametrization of the Second-Class Constraints

Let us now reparametrize the defining set of second-class constraints

$$\Theta^a \quad \longrightarrow \quad \Theta'^a = \Lambda^a{}_b(\Gamma) \, \Theta^b \tag{4.37}$$

in eq. (4.1), and build the Dirac antibracket $(\cdot, \cdot)'_D$ and odd Laplacian Δ'_D from the primed set of constraints Θ'^a. Our aim is dual: First of all, we must check that the different choices of the second-class constraints do not lead to different physical quantities on the physical submanifold. Secondly, it is of interest to know whether a relation can be maintained strongly everywhere in the extended phase space, or whether there appear additional contributions of order $\mathcal{O}(\Theta)$.

The Dirac antibracket does not transform on-shell under reparametrization,

$$(F, G)_D \quad \longrightarrow \quad (F, G)'_D = (F, G)_D + \mathcal{O}(\Theta) . \tag{4.38}$$

In fact, one may calculate the above transformation to any order of precision in Θ. This is important because higher order terms in (4.38) that naively appear to play no physical rôle, can be exposed by Θ-differentiations. The calculations are simplified by choosing coordinates γ^A such that $\Gamma^A = \{\gamma^A; \Theta^a\}$ are transversal coordinates. To second order in Θ one finds

$$(F, G)'_D - (F, G)_D = -(F \frac{\overleftarrow{\partial^r}}{\partial \Theta^a})(\Theta^a \frac{\overleftarrow{\partial^r}}{\partial \Theta'^b})(\Theta'^b, G)_D - (F, \Theta'^a)_D(\frac{\overrightarrow{\partial^l}}{\partial \Theta'^a}\Theta^b)(\frac{\overrightarrow{\partial^l}}{\partial \Theta^b}G)$$

$$-(F, \Theta'^a)_D(\frac{\overrightarrow{\partial^l}}{\partial \Theta'^a}\Theta^b)E_{bc}(\Theta^c \frac{\overleftarrow{\partial^r}}{\partial \Theta'^d})(\Theta'^d, G)_D$$

$$+(F \frac{\overleftarrow{\partial^r}}{\partial \Theta^a})(\Theta^a \frac{\overleftarrow{\partial^r}}{\partial \Theta'^b})(\Theta'^b, \Theta'^c)_D(\frac{\overrightarrow{\partial^l}}{\partial \Theta'^c}\Theta^d)(\frac{\overrightarrow{\partial^l}}{\partial \Theta^d}G) + \mathcal{O}(\Theta^3) . \tag{4.39}$$

In particular, the analogue of eq. (4.5) becomes

$$(\Theta^a, F)'_D = -(\Theta^a \frac{\overleftarrow{\partial^r}}{\partial \Theta'^b})(\Theta'^b, F)_D + (\Theta^a \frac{\overleftarrow{\partial^r}}{\partial \Theta'^b})(\Theta'^b, \Theta'^c)_D(\frac{\overrightarrow{\partial^l}}{\partial \Theta'^c}\Theta^d)(\frac{\overrightarrow{\partial^l}}{\partial \Theta^d}F) + \mathcal{O}(\Theta^3)$$

$$= \mathcal{O}(\Theta) . \tag{4.40}$$

Similarly, to first order in Θ, the Dirac odd Laplacian transforms as

$$
\begin{aligned}
(\Delta_D' F) &= \frac{(-1)^{\epsilon_A}}{2} \frac{\overrightarrow{\partial l}}{\partial \gamma^A} (\gamma^A, F)_D' + \frac{(-1)^{\epsilon_a}}{2} \frac{\overrightarrow{\partial l}}{\partial \Theta^a} (\Theta^a, F)_D' + \frac{1}{2} (\ln \rho_D', F)_D' \\
&= (\Delta_D'' F) - \frac{(-1)^{\epsilon_A}}{2\rho_D''} \frac{\overrightarrow{\partial l}}{\partial \gamma^A} \left[\rho_D'' \left(\gamma^A, \Theta'^a \right)_D \left(\frac{\overrightarrow{\partial l}}{\partial \Theta'^a} \Theta^b \right) \left(\frac{\overrightarrow{\partial l}}{\partial \Theta^b} F \right) \right] \\
&\quad - \frac{(-1)^{\epsilon_A}}{2\rho_D''} \left(\frac{\overrightarrow{\partial l}}{\partial \gamma^A} \Theta'^a \right) \left(\frac{\overrightarrow{\partial l}}{\partial \Theta'^a} \Theta^b \right) \frac{\overrightarrow{\partial l}}{\partial \Theta^b} \left[\rho_D'' (\gamma^A, F)_D \right] \\
&\quad - \frac{(-1)^{\epsilon_a}}{2} \left(\Theta'^a, \frac{\overrightarrow{\partial l}}{\partial \Theta'^a} \Theta^b \right)_D \left(\frac{\overrightarrow{\partial l}}{\partial \Theta^b} F \right) + R(F) ~,
\end{aligned}
\tag{4.41}
$$

where the remainder $R(F) = \mathcal{O}(\Theta^2)$ is a second order differential operator consisting of terms that contain at least as many powers of Θ's as Θ-derivatives. It vanishes to the second order $\mathcal{O}(\Theta^2)$ in Θ when it is normal-ordered. We have furthermore defined

$$
\rho_D'' \equiv \frac{\rho_D'}{\text{sdet}(\frac{\partial \Theta'}{\partial \Theta})} ~,
\tag{4.42}
$$

and

$$
\Delta_D'' \equiv \frac{(-1)^{\epsilon_A}}{2\rho_D''} \frac{\overrightarrow{\partial l}}{\partial \gamma^A} \rho_D'' (\gamma^A, ~\cdot~)_D
\tag{4.43}
$$

is the Dirac odd Laplacian in "unprimed" transversal coordinates $\Gamma^A = \{\gamma^A; \Theta^a\}$ and equipped with ρ_D'' as Dirac measure. To make sure that the Dirac odd Laplacian Δ_D does not transform on-shell,

$$
(\Delta_D F) \quad \longrightarrow \quad (\Delta_D' F) = (\Delta_D F) + \mathcal{O}(\Theta) ~,
\tag{4.44}
$$

we would clearly have to impose $\rho_D'' = \rho_D + \mathcal{O}(\Theta)$, which is just the ρ_D transformation rule (4.10). In general, the Dirac odd Laplacian Δ_D does change when we leave the physical submanifold.

On the other hand, the transformation rule (4.10) provides us with a limited freedom in choosing ρ_D'', or rather, in choosing ρ_D'. It is by construction clear that the squares Δ_D^2, $\Delta_D'^2$ and $\Delta_D''^2$ are all first order differential operators, and let us a priori assume that Δ_D is strongly nilpotent, *i.e.* eq. (4.6). Then the rule $\rho_D'' = \rho_D + \mathcal{O}(\Theta)$ implies that Δ_D'' is at least nilpotent on-shell, because Δ_D does not contain Θ-derivatives (in transversal coordinates). Also Δ_D' becomes nilpotent on-shell,

$$
(\Delta_D^2 F) \quad \longrightarrow \quad (\Delta_D'^2 F) = (\Delta_D^2 F) + O'^A \left(\frac{\overrightarrow{\partial l}}{\partial \Gamma^A} F \right) ~, \qquad O'^A = \mathcal{O}(\Theta) ~,
\tag{4.45}
$$

because each Θ-derivative in (4.41) is accompanied with at least one power of Θ.

The analogue of eq. (4.11), derived using transversal coordinates, becomes

$$
\begin{aligned}
(\Delta_D' \Theta^a) &= -\frac{(-1)^{\epsilon_A}}{2\rho_D} \frac{\overrightarrow{\partial l}}{\partial \gamma^A} \left[\rho_D \left(\gamma^A, \Theta'^b \right)_D \left(\frac{\overrightarrow{\partial l}}{\partial \Theta'^b} \Theta^a \right) \right] - \frac{(-1)^{\epsilon_b}}{2} \left(\Theta'^b, \frac{\overrightarrow{\partial l}}{\partial \Theta'^b} \Theta^a \right)_D + \mathcal{O}(\Theta^2) \\
&= -(\Delta_D \Theta'^b) \left(\frac{\overrightarrow{\partial l}}{\partial \Theta'^b} \Theta^a \right) - (-1)^{\epsilon_b} \left(\Theta'^b, \frac{\overrightarrow{\partial l}}{\partial \Theta'^b} \Theta^a \right)_D + \mathcal{O}(\Theta^2) \\
&= (-1)^{\epsilon_b} \Theta'^b \Delta_D \left(\frac{\overrightarrow{\partial l}}{\partial \Theta'^b} \Theta^a \right) + \mathcal{O}(\Theta^2) = \mathcal{O}(\Theta) ~.
\end{aligned}
\tag{4.46}
$$

Applying Δ'_D one more time one gets

$$(\Delta'^2_D \Theta^a) \;=\; (-1)^{\epsilon_b} \Delta_D(\Theta'^b \Delta_D(\frac{\overrightarrow{\partial^l}}{\partial \Theta'^b}\Theta^a)) + \mathcal{O}(\Theta^2) \;=\; \mathcal{O}(\Theta) \;. \tag{4.47}$$

From this we conclude somewhat surprisingly that Δ'_D is in general *not* nilpotent away from the physical submanifold, independently of the choice of $\rho''_D = \rho_D + \mathcal{O}(\Theta)$. A manifestly reparametrization invariant formulation is to assume the weaker nilpotency (4.36) from the beginning. This has of course no consequences for the physics, which only lives on-shell.

5 Reducible Gauge-Fixing

We consider in this Section an interesting generalization, where the gauge-fixing functions G_{α_0} in the X-part become reducible. This is quite analogous to reducibility among the gauge-generators in the zeroth-level W-part. Recall that originally the stage of reducibility in the W-sector was introduced so that a zeroth-stage gauge theory corresponds to an irreducible gauge algebra, *i.e.* if the ghosts do not carry gauge symmetry. Similarly, first-stage gauge theories have ghosts-for-ghosts that do not carry gauge symmetry, and so forth [2]. We shall here adjust this terminology to the gauge-fixing X-part in the first-level formalism.

The motivation to work with an overcomplete set of constraints is a well-known theme in the theory of constrained dynamics: Often the independent set of constraints breaks symmetries (such as, *e.g.* , Lorentz covariance) or locality that one would like to preserve during the quantization process. Here an overcomplete set of constraints can provide an immediate remedy.

One starts as usual with a zeroth-level theory $W = W(\Gamma; \hbar)$ that has N gauge symmetries that should be fixed. Next one introduces N_0 Lagrange multipliers λ^{α_0} and N_0 antifields $\lambda^*_{\alpha_0}$. For each positive integer i one chooses a number N_i of so-called (first-level) *i'th-stage* ghosts $c^{\alpha_i} \equiv c^{\alpha_i}_{(1)i}$, with Grassmann parity $\epsilon_{\alpha_i} + i$ where $\epsilon_{\alpha_i} \equiv \epsilon^{(1)i}_{\alpha_i}$, and N_i antifields $c^*_{\alpha_i} \equiv c^{i*}_{(1)\alpha_i}$, of opposite statistics, where the index α_i runs through $\alpha_i = 1, \ldots, N_i$. The integers $N_0, N_1, N_2, \ldots$, can be chosen at will, as long as all of the following alternating sums are non-negative:

$$\forall i \geq -1: \quad \sum_{j=-1}^{i} (-1)^{i-j} N_j \;\geq\; 0 \;, \tag{5.1}$$

where $N_{-1} \equiv N$. In particular, one may easily check from the above inequalities that

$$\forall i \geq -1: \quad N_i \;\geq\; 0 \;, \tag{5.2}$$

as it should be. The *stage s of reducibility* is defined as the maximum

$$s \;\equiv\; \max\left(\{i \geq 0 | N_i > 0\} \cup \{-1\}\right) \;, \tag{5.3}$$

over the non-empty set $\{i \geq 0 | N_i > 0\} \cup \{-1\}$. A *zeroth-stage theory* with $s=0$ requires

$$N \;=\; N_0 \;>\; 0 \;=\; N_1 \;=\; N_2 \;=\; N_3 \;=\; \ldots \;. \tag{5.4}$$

This is precisely the irreducible case of Section 3. Similarly, a *first-stage theory* with $s=1$ requires

$$N \;\geq\; 0 \;, \qquad N_0 - N \;=\; N_1 \;>\; 0 \;=\; N_2 \;=\; N_3 \;=\; N_4 \;=\; \ldots \;, \tag{5.5}$$

while a *higher stage theory* requires

$$
\begin{aligned}
s = 2: \quad & N_0 \geq N \geq 0 \,, \quad N_1 - N_0 + N = N_2 > 0 = N_3 = N_4 = N_5 = \dots \,, \\
s = 3: \quad & N_3 - N_2 + N_1 + N = N_0 \geq N \geq 0 \,, \quad N_2 \geq N_3 > 0 = N_4 = N_5 = \dots \,, \\
s = 4: \quad & N_4 - N_3 + N_2 = N_1 - N_0 + N \geq 0 \,, \quad N_0 \geq N \geq 0 \,, \quad N_3 \geq N_4 > 0 = N_5 = \dots \,, \\
& \dots \qquad \dots \,,
\end{aligned}
\tag{5.6}
$$

and so forth. The stage s of reducibility could be ∞. If $s < \infty$, then $\sum_{i=0}^{\infty} N_{2i-1} = \sum_{i=0}^{\infty} N_{2i}$.

Summarizing, the minimal field content in the first-level formalism is

$$
\Gamma^A_{[1]\,\min} \equiv \left\{ \Gamma^A ; \lambda^{\alpha_0}, \lambda^*_{\alpha_0} ; c^{\alpha_1}, c^*_{\alpha_1} ; c^{\alpha_2}, c^*_{\alpha_2} ; \dots ; c^{\alpha_s}, c^*_{\alpha_s} \right\} \,.
\tag{5.7}
$$

In addition to the $2\sum_{i=-1}^{s} N_i$ minimal fields and antifields, there is a triangular tower of $4\sum_{i=1}^{s} i N_i$ non-minimal fields and antifields, in complete analogy with reducibility at the zeroth-level, cf. Ref. [2] and Subsection 5.7. The first-level minimal odd Laplacian becomes

$$
\Delta_{[1]\,\min} \equiv \Delta + (-1)^{\epsilon_{\alpha_0}} \frac{\overrightarrow{\partial^l}}{\partial \lambda^{\alpha_0}} \frac{\overrightarrow{\partial^l}}{\partial \lambda^*_{\alpha_0}} + \sum_{i=1}^{s} (-1)^{\epsilon_{\alpha_i}+i} \frac{\overrightarrow{\partial^l}}{\partial c^{\alpha_i}} \frac{\overrightarrow{\partial^l}}{\partial c^*_{\alpha_i}} \,,
\tag{5.8}
$$

while the first-level minimal Planck-number operator is

$$
\mathsf{Pl}_{\min} = -\left(\lambda^*_{\alpha_0} \lambda^{\alpha_0} + \sum_{i=1}^{s} (i+1) c^*_{\alpha_i} c^{\alpha_i}, \ \cdot \ \right)_{[1]} + \hbar_{(1)} \frac{\partial}{\partial \hbar_{(1)}} \,.
\tag{5.9}
$$

The gauge-fixing action $X_{\min}$ should again satisfy the Principle 3.1, *i.e.* 1) Planck number conservation, 2) the Quantum Master Equation and 3) rank requirements. Although it is straightforward to expand $X_{\min}$ in action terms allowed by Planck number, it quickly becomes space consuming. Instead we shall focus on a few important terms

$$
\begin{aligned}
X_{\min} = \ & G_{\alpha_0}\, \lambda^{\alpha_0} + (i\hbar_{(1)})\, H + \lambda^*_{\alpha_0}\, Z^{\alpha_0}{}_{\alpha_1}\, c^{\alpha_1} + \sum_{i=1}^{s-1} c^*_{\alpha_i}\, Z^{\alpha_i}{}_{\alpha_{i+1}}\, c^{\alpha_{i+1}} \\
& + \lambda^*_{\alpha_0} \left[\frac{1}{2} U^{\alpha_0}_{\beta_0\gamma_0}\, \lambda^{\gamma_0} (-1)^{\epsilon_{\beta_0}+1} + (i\hbar_{(1)})\, V^{\alpha_0}{}_{\beta_0} \right] \lambda^{\beta_0} + (i\hbar_{(1)})^2\, \lambda^*_{\alpha_0} \tilde{G}^{\alpha_0} \\
& + \sum_{i=1}^{s} c^*_{\alpha_i} \left[U^{\alpha_i}_{\beta_i\alpha_0}\, \lambda^{\alpha_0} (-1)^{\epsilon_{\beta_i}+i+1} + (i\hbar_{(1)})\, V^{\alpha_i}{}_{\beta_i} \right] c^{\beta_i} + \dots \,.
\end{aligned}
\tag{5.10}
$$

In particular, the Faddeev-Popov term $\lambda^*_{\alpha_0} Z^{\alpha_0}{}_{\alpha_1} c^{\alpha_1}$ and its higher-stage counterparts $c^*_{\alpha_i} Z^{\alpha_i}{}_{\alpha_{i+1}} c^{\alpha_{i+1}}$ will be important new ingredients (as compared to the irreducible case). The structure functions $Z^{\alpha_{i-1}}{}_{\alpha_i} \equiv Z^{\alpha_{i-1}}_i{}_{\alpha_i}(\Gamma;\hbar)$ will carry Grassmann parity $\epsilon_{\alpha_i} + \epsilon_{\alpha_{i-1}}$. We stress that the eq. (5.10) should not be read as a systematic expansion of the action $X_{\min}$. Rather the terms in eq. (5.10) were selected simply because they enter the first few consequences of the Quantum Master Equation for $X_{\min}$,

$$
\begin{aligned}
(G_{\alpha_0}, G_{\beta_0}) &= G_{\gamma_0} U^{\gamma_0}_{\alpha_0\beta_0} \,, & (5.11) \\
G_{\alpha_0} Z^{\alpha_0}{}_{\alpha_1} &= 0 \,, & (5.12) \\
Z^{\alpha_{i-1}}{}_{\alpha_i} Z^{\alpha_i}{}_{\alpha_{i+1}} &= \mathcal{O}(G) \,, & (5.13) \\
(Z^{\alpha_i}{}_{\beta_{i+1}}, G_{\beta_0}) &= Z^{\alpha_i}{}_{\alpha_{i+1}} U^{\alpha_{i+1}}_{\beta_{i+1}\beta_0} - (-1)^{(\epsilon_{\beta_0}+1)(\epsilon_{\gamma_i}+\epsilon_{\beta_{i+1}})} U^{\alpha_i}_{\gamma_i\beta_0} Z^{\gamma_i}{}_{\beta_{i+1}} + \mathcal{O}(G) \,, & (5.14) \\
(\Delta G_{\beta_0}) - (H, G_{\beta_0}) &= \sum_{i=0}^{s} (-1)^{\epsilon_{\alpha_i}+i} U^{\alpha_i}_{\alpha_i\beta_0} + G_{\alpha_0} V^{\alpha_0}{}_{\beta_0} \,, & (5.15) \\
-(\Delta H) + \frac{1}{2}(H, H) &= \sum_{i=0}^{s} V^{\alpha_i}{}_{\alpha_i} - G_{\alpha_0} \tilde{G}^{\alpha_0} \,. & (5.16)
\end{aligned}
$$

The first eq. (5.11) is just the usual non-Abelian involution of the gauge-fixing functions G_{α_0}. The second eq. (5.12) and the third eq. (5.13) show that G_{α_0} and $Z^{\alpha_{i-1}}{}_{\alpha_i}$, respectively, are in general reducible. They imply that the action X exhibit gauge symmetries on-shell with respect to the G_{α_0}'s,

$$\delta c^{\alpha_i} = Z^{\alpha_i}{}_{\alpha_{i+1}} \xi^{\alpha_{i+1}} , \tag{5.17}$$

where $\xi^{\alpha_{i+1}}$ are gauge parameters.

Note that an irreducible gauge-fixing action X corresponding to $s = 0$ has *no* gauge symmetry at the first-level[**]. This is why we could choose a trivial first-level gauge $\Psi \equiv 0$ in Subsection 3.3. In the reducible case the gauge fermion Ψ should meet certain rank requirements.

5.1 First-Stage Reducibility

In Subsections 5.1-5.6 we shall work out the simplest case of reducible gauge-fixing in detail, namely first-stage reducibility. In this setting the gauge-fixing action X has N_1 gauge symmetries in the λ^{α_0}-variables due to reducibility eq. (5.12) among the gauge-fixing functions G_{α_0}. Besides the two minimal first-level fields λ^{α_0} and c^{α_1}, there are also two non-minimal fields, $\bar{c}_{\alpha_1}$ of statistics $\epsilon_{\alpha_1} + 1$, and π_{α_1} of statistics ϵ_{α_1}, *i.e.*

$$\Gamma^A_{[1]} \equiv \left\{ \Gamma^A ; \lambda^{\alpha_0}, \lambda^*_{\alpha_0} ; c^{\alpha_1}, c^*_{\alpha_1} ; \bar{c}_{\alpha_1}, \bar{c}^{*\alpha_1} ; \pi_{\alpha_1}, \pi^{*\alpha_1} \right\} . \tag{5.18}$$

The Planck-number operator is chosen to be

$$\mathsf{Pl} = -\left(\lambda^*_{\alpha_0} \lambda^{\alpha_0} + 2 c^*_{\alpha_1} c^{\alpha_1} - \bar{c}_{\alpha_1} \bar{c}^{*\alpha_1}, \cdot \right)_{[1]} + \hbar_{(1)} \frac{\partial}{\partial \hbar_{(1)}} , \tag{5.19}$$

or equivalently,

$$\mathsf{Pl}(\lambda^{\alpha_0}) = 1 \quad , \quad \mathsf{Pl}(c^{\alpha_1}) = 2 \quad , \quad \mathsf{Pl}(\bar{c}_{\alpha_1}) = -1 \quad , \quad \mathsf{Pl}(\pi_{\alpha_1}) = 0 ,$$

$$\mathsf{Pl}(\lambda^*_{\alpha_0}) = -1 \quad , \quad \mathsf{Pl}(c^*_{\alpha_1}) = -2 \quad , \quad \mathsf{Pl}(\bar{c}^{*\alpha_1}) = 1 \quad , \quad \mathsf{Pl}(\pi^{*\alpha_1}) = 0 , \tag{5.20}$$

$$\mathsf{Pl}(\Gamma^A) = 0 \quad , \qquad\qquad\qquad \mathsf{Pl}(\hbar_{(1)}) = 1 \quad , \quad \mathsf{Pl}(\hbar) = 0 .$$

Note in particular that π_{α_1} and $\pi^{*\alpha_1}$ have vanishing Planck number, so they appear on the same footing as the original variables Γ^A in a Planck number expansion. This is just one of many reasons to enlarge the $2N$-dimensional zeroth-level phase space Γ^A into a $2N_0$-dimensional phase space

$$\Gamma^{A_{\text{ext}}} \equiv \{ \Gamma^A ; \pi_{\alpha_1}, \pi^{*\alpha_1} \} , \tag{5.21}$$

where $N_0 = N + N_1$. We shall see in Subsection 5.4 that this space plays a profound rôle. The odd Laplacian reads

$$\Delta_{[1]} \equiv \Delta_{\text{ext}} + (-1)^{\epsilon_{\alpha_0}} \frac{\overrightarrow{\partial^l}}{\partial \lambda^{\alpha_0}} \frac{\overrightarrow{\partial^l}}{\partial \lambda^*_{\alpha_0}} + (-1)^{\epsilon_{\alpha_1}+1} \left(\frac{\overrightarrow{\partial^l}}{\partial c^{\alpha_1}} \frac{\overrightarrow{\partial^l}}{\partial c^*_{\alpha_1}} + \frac{\overrightarrow{\partial^l}}{\partial \bar{c}_{\alpha_1}} \frac{\overrightarrow{\partial^l}}{\partial \bar{c}^{*\alpha_1}} \right) , \tag{5.22}$$

where

$$\Delta_{\text{ext}} \equiv \Delta + (-1)^{\epsilon_{\alpha_1}} \frac{\overrightarrow{\partial^l}}{\partial \pi_{\alpha_1}} \frac{\overrightarrow{\partial^l}}{\partial \pi^{*\alpha_1}} . \tag{5.23}$$

[**]Compare this with the zeroth-level terminology, where a *trivial* gauge algebra, *i.e.* with *no* gauge symmetry, is strictly speaking of stage "−1". In other words, the definition of stage of the first-level X-action has been shifted by one unit as compared to the original definition of stage of the zeroth-level W-action [2]. This shift is introduced to avoid speaking of negative stages.

The subscript "ext" will everywhere in this Section refer to the extended space $\Gamma^{A_{\text{ext}}}$.

However let us first take a more traditional route. Recall that in the original field-antifield approach the minimal sector is introduced to obtain solutions to the Master Equation satisfying the appropriate rank condition, and the non-minimal sector is *only* added to have well-defined gauge-fixing choices at hand [1, 2]. So ignoring for the moment the non-minimal fields $\{\bar{c}_{\alpha_1}, \bar{c}^{*\alpha_1}; \pi_{\alpha_1}, \pi^{*\alpha_1}\}$, the rank of the Hessian of $X_{\min}$ in the minimal sector

$$\Gamma^A_{[1]\min} \equiv \{\Gamma^A; \lambda^{\alpha_0}, \lambda^*_{\alpha_0}; c^{\alpha_1}, c^*_{\alpha_1}\} \tag{5.24}$$

should be $2N_0$, where $N_0 = N + N_1$. This implies that the two rectangular matrices $\partial G_{\alpha_0}/\partial\Gamma^A$ and $Z^{\alpha_0}{}_{\alpha_1}$ have maximal rank, *i.e.*

$$\text{rank}(G_{\alpha_0}\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A}) = N , \qquad\qquad \text{rank}(Z^{\alpha_0}{}_{\alpha_1}) = N_1 , \tag{5.25}$$

and therefore the $Z^{\alpha_0}{}_{\alpha_1}$ matrix does not have zero-eigenvalue right eigenvectors.

5.2 First-Level Gauge-Fixing

The standard Ansatz for the non-minimal action is

$$X = X_{\min}(\Gamma^A_{[1]\min}; \hbar_{[1]}) + \pi_{\alpha_1}\bar{c}^{*\alpha_1} , \tag{5.26}$$

while a simple choice for Ψ reads

$$\Psi = -\bar{c}_{\alpha_1}\chi^{\alpha_1} , \qquad\qquad \text{Pl}(\Psi) = 0 , \tag{5.27}$$

with N_1 first-level gauge-fixing conditions $\chi^{\alpha_1} = \chi^{\alpha_1}(\Gamma; \lambda_0; \hbar)$ of Grassmann parity ϵ_{α_1}. Planck number conservation restricts us to a linear dependence of λ^{α_0},

$$\chi^{\alpha_1} = \omega^{\alpha_1}{}_{\alpha_0}\lambda^{\alpha_0} \tag{5.28}$$

(up to an inessential constant proportional to $\hbar_{(1)}$). This gauge-fixing choice is based on a matrix $\omega^{\alpha_1}{}_{\alpha_0} = \omega^{\alpha_1}{}_{\alpha_0}(\Gamma; \hbar)$ of Grassmann parity $\epsilon_{\alpha_0}+\epsilon_{\alpha_1}$, such that the Faddeev-Popov matrix

$$\Delta^{\alpha_1}{}_{\beta_1} \equiv \omega^{\alpha_1}{}_{\alpha_0}Z^{\alpha_0}{}_{\beta_1} \tag{5.29}$$

is invertible, *i.e.*

$$\text{rank}(\Delta^{\alpha_1}{}_{\beta_1}) = N_1 . \tag{5.30}$$

One could in principle let $\omega^{\alpha_1}{}_{\alpha_0}$ depend on π_{α_1}, but one may prove a constraint $\pi_{\alpha_1} \approx 0$ that would turn this idea into a vacuous exercise. According to the general theory outlined in Subsection 3.2 the first-level partition function is given by

$$\mathcal{Z}^\Psi_{[1]} = \int d\mu \, e^{\frac{i}{\hbar}(W+X^\Psi)}\Big|_{\lambda^*_0, c^*_1, \bar{c}^*_1, \pi^*_1=0} = \int d\mu \, e^{\frac{i}{\hbar}(W^\Psi+X)}\Big|_\Sigma , \tag{5.31}$$

with a measure

$$d\mu = \rho[d\Gamma][d\lambda_0][dc_1][d\bar{c}_1][d\pi_1] , \tag{5.32}$$

and a gauge-fixing surface Σ specified by

$$\lambda_{\alpha_0}^* \;\; = \;\; -E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\lambda^{\alpha_0}} \;\; = \;\; \bar{c}_{\alpha_1}E(\mathrm{ad}\Psi)\omega^{\alpha_1}{}_{\alpha_0} \;, \tag{5.33}$$

$$c_{\alpha_1}^* \;\; = \;\; -E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial c^{\alpha_1}} \;\; = \;\; 0 \;, \tag{5.34}$$

$$\bar{c}^{*\alpha_1} \;\; = \;\; -E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\bar{c}_{\alpha_1}} \;\; = \;\; E(\mathrm{ad}\Psi)\chi^{\alpha_1} \;, \tag{5.35}$$

$$\pi^{*\alpha_1} \;\; = \;\; -E(\mathrm{ad}\Psi)\frac{\partial\Psi}{\partial\pi_{\alpha_1}} \;\; = \;\; 0 \;, \tag{5.36}$$

cf. the prescription (3.28).

Lemma 5.1 :   *The path integrations over the Faddeev-Popov ghost pair $\{c^{\alpha_1};\bar{c}_{\alpha_1}\}$ can be performed explicitly. The first-level partition function (5.31) thereby simplifies to*

$$\mathcal{Z}_{[1]} \;\; = \;\; \int[d\Gamma][d\lambda_0][d\pi_1]\,\rho\,e^{\frac{i}{\hbar}W+\frac{i}{\hbar}G_{\omega,\alpha_0}\lambda^{\alpha_0}-H}\,\mathrm{sdet}(\Delta^{\alpha_1}{}_{\beta_1}) \;, \tag{5.37}$$

where

$$G_{\omega,\alpha_0} \;\; \equiv \;\; G_{\alpha_0} + \pi_{\alpha_1}\omega^{\alpha_1}{}_{\alpha_0} \tag{5.38}$$

are ω-deformed gauge-fixing constraints, and $\Delta^{\alpha_1}{}_{\beta_1}$ is the Faddeev-Popov matrix (5.29).

PROOF OF LEMMA 5.1: It is convenient to split the gauge-fixed action

$$S \;\; = \;\; (\frac{\hbar_{(1)}}{\hbar}W^{\Psi}+X)|_{\Sigma} \;\; = \;\; S_0+S_{FP}+V \;, \tag{5.39}$$

into a part

$$S_0 \;\; \equiv \;\; \frac{\hbar_{(1)}}{\hbar}W + G_{\alpha_0}\lambda^{\alpha_0} + (i\hbar_{(1)})H + \pi_{\alpha}\chi^{\alpha} \tag{5.40}$$

that is independent of the ghosts and antighosts $\{c^{\alpha_1};\bar{c}_{\alpha_1}\}$, a Faddeev-Popov term $S_{FP} \equiv \bar{c}_{\alpha_1}\Delta^{\alpha_1}{}_{\beta_1}c^{\beta_1}$ that is quadratic in $\{c^{\alpha_1};\bar{c}_{\alpha_1}\}$, and a part V that contains all interaction terms, tadpole terms and terms quadratic in the antighost $\bar{c}_{\alpha_1}$. At this point the only quantities left that carry Planck number, are $\hbar_{(1)}$, λ^{α_0}, c^{α_1} and $\bar{c}_{\alpha_1}$. Since the action S has Planck number $\mathsf{Pl}(S) = 1$, the multiplicities $m_{\hbar}$, m_{λ}, m_1, and $\bar{m}_1$ of $\hbar_{(1)}$, λ^{α_0}, c^{α_1} and $\bar{c}_{\alpha_1}$, respectively, must obey

$$m_{\hbar} + m_{\lambda} + 2m_1 - \bar{m}_1 \;\; = \;\; 1 \tag{5.41}$$

in any given term in the action. Equivalently,

$$\bar{m}_1 - m_1 \;\; = \;\; m_{\hbar} + m_{\lambda} + m_1 - 1 \;\; \equiv \;\; RHS \;. \tag{5.42}$$

Clearly the right hand side $RHS \geq -1$. There are no terms with $RHS{=}{-}1$, and the terms with $RHS{=}0$ are precisely the free part S_0+S_{FP}. This implies that all the terms in the V-part have fewer ghosts c^{α_1} than antighosts $\bar{c}_{\alpha_1}$, i.e.

$$m_1 \;\; < \;\; \bar{m}_1 \;. \tag{5.43}$$

Next one scales the ghosts and antighosts,

$$c^{\alpha_1} \;\; \rightarrow \;\; \frac{1}{\varepsilon}c^{\alpha_1} \;, \qquad\qquad \bar{c}_{\alpha_1} \;\; \rightarrow \;\; \varepsilon\bar{c}_{\alpha_1} \;, \tag{5.44}$$

and let $\varepsilon \to 0$. The V-term drops out because of the rule (5.43), while the free part S_0+S_{FP} and the path integral measure are unchanged. Hence the integration over $\{c^{\alpha_1};\bar{c}_{\alpha_1}\}$ can be explicitly performed.

$\square$

5.3   A Square Root Formula for H

The ω-deformed constraints G_{ω,α_0} have two very important properties:

1. First, the N_0 constraints G_{ω,α_0} are *irreducible* on the $2N_0$-dimensional extended space

$$\Gamma^{A_{\text{ext}}} \equiv \{\Gamma^A; \pi_{\alpha_1}, \pi^{*\alpha_1}\} \ . \tag{5.45}$$

Besides containing the original reducible constraints $G_{\alpha_0} \approx 0$ of rank N, the ω-deformed constraints $G_{\omega,\alpha_0} \approx 0$ in addition contain N_1 conditions $\pi_{\alpha_1} \approx 0$, as is clear from the formula

$$\pi_{\beta_1} = G_{\omega,\alpha_0} Z^{\alpha_0}{}_{\alpha_1} (\Delta^{-1})^{\alpha_1}{}_{\beta_1} \ , \tag{5.46}$$

where use has been made of eq. (5.12). See Subsection 5.5 below for more details.

2. Second, the G_{ω,α_0} are in non-Abelian involution

$$(G_{\omega,\alpha_0}, G_{\omega,\beta_0})_{\text{ext}} = G_{\omega,\gamma_0} U^{\gamma_0}_{\omega,\alpha_0\beta_0} \ , \tag{5.47}$$

with respect to the extended $(\cdot,\cdot)_{\text{ext}}$ bracket. The ω-deformed structure functions read

$$
\begin{aligned}
U^{\gamma_0}_{\omega,\alpha_0\beta_0} = {} & \frac{1}{2}\left[\delta^{\gamma_0}_{\delta_0} - Z^{\gamma_0}{}_{\alpha_1}(\Delta^{-1})^{\alpha_1}{}_{\beta_1}\omega^{\beta_1}{}_{\delta_0}\right] U^{\delta_0}_{\alpha_0\beta_0} + Z^{\gamma_0}{}_{\alpha_1}(\Delta^{-1})^{\alpha_1}{}_{\beta_1}(\omega^{\beta_1}{}_{\alpha_0}, G_{\omega,\beta_0})_{\text{ext}} \\
& -(-1)^{(\epsilon_{\alpha_0}+1)(\epsilon_{\beta_0}+1)}(\alpha_0 \leftrightarrow \beta_0) + \mathcal{O}(G_\omega)
\end{aligned}
\tag{5.48}
$$

to the zeroth order in G_ω.

In the following Theorem 5.2 the fundamental rôle played by the $2N_0$-dimensional extended space $\Gamma^{A_{\text{ext}}}$ is displayed further. Let

$$F^{\alpha_0}_{\bar{\omega}} \equiv F^{\alpha_0} + \bar{\omega}^{\alpha_0}{}_{\alpha_1}\pi^{*\alpha_1} + \tilde{\omega}^{\alpha_0\alpha_1}\pi_{\alpha_1} + \ldots \tag{5.49}$$

be arbitrary zeroth-level coordinate functions of statistics $\epsilon_{\alpha_0}+1$, where "$\ldots$" denotes terms that are at least quadratic in π_{α_1} and $\pi^{*\alpha_1}$, and where

$$F^{\alpha_0} = F^{\alpha_0}(\Gamma;\hbar) \ , \qquad \bar{\omega}^{\alpha_0}{}_{\alpha_1} = \bar{\omega}^{\alpha_0}{}_{\alpha_1}(\Gamma;\hbar) \qquad \text{and} \qquad \tilde{\omega}^{\alpha_0\alpha_1} = \tilde{\omega}^{\alpha_0\alpha_1}(\Gamma;\hbar) \ , \tag{5.50}$$

such that the transformation

$$\Gamma^{A_{\text{ext}}} \equiv \{\Gamma^A; \pi_{\alpha_1}, \pi^{*\alpha_1}\} \quad \longrightarrow \quad \bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}} \equiv \{F^{\alpha_0}_{\bar{\omega}}; G_{\omega,\alpha_0}\} \tag{5.51}$$

is a coordinate transformation of the $2N_0$-dimensional space $\Gamma^{A_{\text{ext}}}$, and let $J_{\omega,\bar{\omega}} = \text{sdet}(\frac{\partial \bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}}}{\partial \Gamma^{B_{\text{ext}}}})$ denote the Jacobian. Then

Theorem 5.2 : *The Quantum Master Equation for* $X_{\min}$ *implies that the quantum correction H is given the following square root formula*

$$H = -\ln\sqrt{\left.\frac{J_{\omega,\bar{\omega}}\ \text{sdet}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}}}{\rho\ \text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})^2}\right|_{\pi_1,\pi_1^*=0}} + \mathcal{O}(G) \ . \tag{5.52}$$

There are essentially two new features in the reducible case that we would like to emphasize as compared to the irreducible case, cf. Theorem 3.2. First and most pronounced, almost all reference to the original space Γ^A has been replaced with the extended space $\Gamma^{A_{\text{ext}}}$. The two conditions $G_{\alpha_0} \approx 0$ and $\pi_{\alpha_1} \approx 0$ enter slightly asymmetrically, because $H = H(\Gamma; \hbar)$ conventionally does not depend on π_{α_1}, and hence this asymmetry is due to notation rather than substance, cf. Subsection 5.4. Secondly, the Faddeev-Popov determinant $\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})$ makes an interesting appearance in formula (5.52).

In more details the Jacobian is given as

$$
J_{\omega,\bar{\omega}} \;\equiv\; \text{sdet}(\bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}}\frac{\overleftarrow{\partial^r}}{\partial\Gamma^{B_{\text{ext}}}}) \;=\; \int[d\bar{C}_{\text{ext}}][dC_{\text{ext}}] \; e^{\frac{i}{\hbar}S_C} \;, \tag{5.53}
$$

with a Jacobian action

$$
\begin{aligned}
S_C \;&=\; \bar{C}_{A_{\text{ext}}}(\bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}}\frac{\overleftarrow{\partial^r}}{\partial\Gamma^{B_{\text{ext}}}})C^{B_{\text{ext}}} \\
&=\; (-1)^{\epsilon_{A_{\text{ext}}}+1}C^{A_{\text{ext}}}(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^{A_{\text{ext}}}}G_{\omega,\alpha_0})\bar{C}^{\alpha_0} + \bar{C}_{\alpha_0}(F^{\alpha_0}_{\bar{\omega}}\frac{\overleftarrow{\partial^r}}{\partial\Gamma^{A_{\text{ext}}}})C^{A_{\text{ext}}} \\
&=\; \left[(-1)^{\epsilon_A+1}C^A(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}G_{\alpha_0}) + (-1)^{\epsilon_{\alpha_1}+1}C_{\alpha_1}\omega^{\alpha_1}{}_{\alpha_0}\right]\bar{C}^{\alpha_0} \\
&\quad +\bar{C}_{\alpha_0}\left[(F^{\alpha_0}\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A + \bar{\omega}^{\alpha_0}{}_{\alpha_1}C^{\alpha_1} + \tilde{\omega}^{\alpha_0\alpha_1}C_{\alpha_1}\right] + \mathcal{O}(\pi_1; \pi_1^*) \;. \tag{5.54}
\end{aligned}
$$

Similarly, one may exponentiate the other superdeterminant

$$
\text{sdet}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}} \;=\; \int[d\bar{B}_0][dB_0] \; e^{\frac{i}{\hbar}S_B} \tag{5.55}
$$

with action

$$
\begin{aligned}
S_B \;&=\; \bar{B}_{\alpha_0}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}}B^{\beta_0} \\
&=\; \bar{B}_{\alpha_0}\left[(F^{\alpha_0}, G_{\beta_0}) - \bar{\omega}^{\alpha_0}{}_{\alpha_1}\omega^{\alpha_1}{}_{\beta_0}\right]B^{\beta_0} + \mathcal{O}(\pi_1; \pi_1^*) \;. \tag{5.56}
\end{aligned}
$$

Here the ghost pair $(C^{A_{\text{ext}}}; \bar{C}_{A_{\text{ext}}})$ has statistics $\epsilon_{A_{\text{ext}}}+1$ and the ghost pair $(B^{\alpha_0}; \bar{B}_{\alpha_0})$ has statistics $\epsilon_{\alpha_0}+1$. We have split $\bar{C}_{A_{\text{ext}}} = \{\bar{C}_{\alpha_0}; \bar{C}^{\alpha_0}\}$ of Grassmann parity $\epsilon(\bar{C}_{\alpha_0}) = \epsilon_{\alpha_0}$ and $\epsilon(\bar{C}^{\alpha_0}) = \epsilon_{\alpha_0}+1$, respectively. The ghosts $C^{A_{\text{ext}}}$ of the $2N_0$-dimensional extended space decompose as $C^{A_{\text{ext}}} = \{C^A; C_{\alpha_1}, C^{\alpha_1}\}$, with Grassmann parity $\epsilon(C^A) = \epsilon_A+1$, $\epsilon(C_{\alpha_1}) = \epsilon_{\alpha_1}+1$ and $\epsilon(C^{\alpha_1}) = \epsilon_{\alpha_1}$, respectively.

Due to the above mentioned properties of G_{ω,α_0}, it is clear that one may re-use the Lemma 3.3 from Section 3.3 for this situation:

Lemma 5.3 : *The factor $J_{\omega,\bar{\omega}}\,\text{sdet}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}}$ is independent of $F^{\alpha_0}_{\bar{\omega}}$ up to terms that vanish on-shell with respect to G_{ω,α_0}, if the G_{ω,α_0}'s are in involution with respect to the $(\cdot,\cdot)_{\text{ext}}$ bracket.*

PROOF OF THEOREM 5.2: We would like to derive an analogue of eq. (3.18) with G_α replaced with the ω-deformed G_{ω,α_0} in the extended space $\Gamma^{A_{\text{ext}}}$. From the Master Equation, we are provided with a reducible version (5.15). Let us first contract an index on the ω-deformed structure functions (5.48),

$$
\begin{aligned}
(-1)^{\epsilon_{\alpha_0}}U^{\alpha_0}_{\omega,\alpha_0\beta_0} \;&=\; (-1)^{\epsilon_{\alpha_0}}\left[\delta^{\alpha_0}_{\gamma_0} - Z^{\alpha_0}{}_{\alpha_1}(\Delta^{-1})^{\alpha_1}{}_{\beta_1}\omega^{\beta_1}{}_{\gamma_0}\right]U^{\gamma_0}_{\alpha_0\beta_0} \\
&\quad +(-1)^{\epsilon_{\alpha_0}}Z^{\alpha_0}{}_{\alpha_1}(\Delta^{-1})^{\alpha_1}{}_{\beta_1}(\omega^{\beta_1}{}_{\alpha_0}, G_{\omega,\beta_0})_{\text{ext}} + \mathcal{O}(G_\omega) \;, \tag{5.57}
\end{aligned}
$$

where use has been made of eq. (5.12). Next one derives

$$(-1)^{\epsilon_{\alpha_1}} U^{\alpha_1}_{\alpha_1\beta_0} = (-1)^{\epsilon_{\alpha_1}} (\Delta^{-1})^{\alpha_1}{}_{\beta_1} \omega^{\beta_1}{}_{\alpha_0} (Z^{\alpha_0}{}_{\alpha_1}, G_{\omega,\beta_0})_{\text{ext}}$$
$$+ (-1)^{\epsilon_{\alpha_0}} Z^{\alpha_0}{}_{\alpha_1} (\Delta^{-1})^{\alpha_1}{}_{\beta_1} \omega^{\beta_1}{}_{\gamma_0} U^{\gamma_0}_{\alpha_0\beta_0} + \mathcal{O}(G_\omega) \tag{5.58}$$

from eq. (5.14). Combining equation (5.15), (5.57) and (5.58), one gets the sought-for equation

$$(\Delta_{\text{ext}} G_{\omega,\beta_0}) - (H - \ln\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1}), \ G_{\omega,\beta_0})_{\text{ext}} = (-1)^{\epsilon_{\alpha_0}} U^{\alpha_0}_{\omega,\alpha_0\beta_0} + \mathcal{O}(G_\omega) \ . \tag{5.59}$$

Besides working in the extended space $\Gamma^{A_{\text{ext}}}$, the only real difference from eq. (3.18) is that H has been shifted by the Faddeev-Popov determinant. Hence one may proceed as in the proof of Theorem 3.2.

$\square$

5.4   From Irreducible to Reducible Constraints

The partition function for reducible gauge-fixing constraints becomes

$$\mathcal{Z}^{G_\omega}_{[1]} = \int [d\Gamma][d\pi_1] \ e^{\frac{i}{\hbar}W} \ \delta(G_{\omega,\alpha_0}) \sqrt{\rho \ J_{\omega,\bar{\omega}} \ \text{sdet}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}}} \Big|_{\pi_1^* = 0} \tag{5.60}$$

by combining the two previous results (5.37) and (5.52). Note that the Faddeev-Popov determinant have completely cancelled out from the partition function (5.60)! This suggests that there is a much simpler and broader approach as follows: If you want N_0 reducible gauge-fixing constraints in the original $2N$-dimensional zeroth-level phase space Γ^A, then introduce first-level variable $\{\lambda^{\alpha_0}, \lambda^*_{\alpha_0}; \pi_{\alpha_1}, \pi^{*\alpha_1}\}$, and an action

$$X = \mathcal{G}_{\alpha_0} \lambda^{\alpha_0} + (i\hbar_{(1)})\mathcal{H} - \lambda^*_{\alpha_0} \mathcal{R}^{\alpha_0} + \mathcal{O}\left((\lambda^*)^2\right), \tag{5.61}$$

where all the structure functions $\mathcal{G}_{\alpha_0} = \mathcal{G}_{\alpha_0}(\Gamma_{\text{ext}}; \hbar)$, $\mathcal{H} = \mathcal{H}(\Gamma_{\text{ext}}; \hbar)$, etc, are allowed to depend on π_{α_1} and $\pi^{*\alpha_1}$ as well. Hence the full constraints $\mathcal{G}_{\alpha_0}$ live in the extended $\Gamma^{A_{\text{ext}}}$ space and should be irreducible, because of rank requirements, while the reducible constraints are simply the restriction

$$G_{\alpha_0} = \mathcal{G}_{\alpha_0}\big|_{\pi_1, \pi_1^* = 0} \tag{5.62}$$

into the original Γ^A space. According to Theorem 3.2 applied on the extended space $\Gamma^{A_{\text{ext}}}$, the Quantum Master Equation for X will carve out the square root measure factor directly,

$$\mathcal{H} = -\ln\sqrt{\frac{\mathcal{J} \ \text{sdet}(\mathcal{F}^{\alpha_0}, \mathcal{G}_{\beta_0})_{\text{ext}}}{\rho}} + \mathcal{O}(\mathcal{G}) \ , \tag{5.63}$$

with $\mathcal{J} \equiv \text{sdet}(\frac{\partial \bar{\Gamma}^{A_{\text{ext}}}}{\partial \Gamma^{B_{\text{ext}}}})$ and $\bar{\Gamma}^{A_{\text{ext}}} \equiv \{\mathcal{F}^{\alpha_0}; \mathcal{G}_{\alpha_0}\}$. There is thus no need to introduce a Faddeev-Popov ghost pair $\{c^{\alpha_1}; \bar{c}_{\alpha_1}\}$. Therefore by appealing to the first-level irreducible theory from Subsection 3.2-3.3, one derives the following version of Corollary 3.5, written in the so-called $\lambda^*_{\alpha_0} = 0 = \pi^{*\alpha_1}$ gauge.

Corollary 5.4 : *The partition function*

$$\mathcal{Z}^{\mathcal{G}}_{[1]} = \int [d\Gamma_{\text{ext}}] \ e^{\frac{i}{\hbar}W} \ \delta(\pi^{*\alpha_1}) \ \delta(\mathcal{G}_{\alpha_0}) \sqrt{\rho \ \mathcal{J} \ \text{sdet}(\mathcal{F}^{\alpha_0}, \mathcal{G}_{\beta_0})_{\text{ext}}} \tag{5.64}$$

is independent of $\mathcal{G}_{\alpha_0}$'s that are in involution with respect to the $(\cdot, \cdot)_{\text{ext}}$-bracket.

We conclude the following

Theorem 5.5 – Reduction Theorem: *The irreducible partition function (5.64) on the $2N_0$-dimensional extended space $\Gamma^{A_{\text{ext}}}$ reduces to the reducible partition function (5.60) on the $2N$-dimensional original space Γ^A, if one chooses the gauge-fixing conditions $\mathcal{G}_{\alpha_0}$ to be the ω-deformed constraints $G_{\omega,\alpha_0} \equiv G_{\alpha_0} + \pi_{\alpha_1}\omega^{\alpha_1}{}_{\alpha_0}$.*

In the next Subsection we will present a Reduction Theorem 5.6 that in some respect is opposite to the above Reduction Theorem 5.5.

5.5 From Reducible to Irreducible Constraints

One may always assume that the reducible constraints G_{α_0}, $\alpha_0 = 1, \ldots, N_0$, can be written as linear combinations of irreducible constraints G'_α, $\alpha = 1, \ldots, N$,

$$G_{\alpha_0} = G'_\alpha P^\alpha{}_{\alpha_0}, \tag{5.65}$$

such that the irreducible constraints G'_α are in involution, cf. eq. (3.17). Therefore the theory can be set up purely within the irreducible framework of Section 3. In this Subsection we check that the reducible and the irreducible approach agree.

Theorem 5.6 – Reduction Theorem: *The reducible partition function (5.60) coincides with the irreducible partition function (3.49), when re-writing the reducible quantities in their irreducible counterparts.*

PROOF: First of all, let us note that the reducible gradients

$$\left(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}G_{\alpha_0}\right) = \left(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}G'_\alpha\right)P^\alpha{}_{\alpha_0} + \mathcal{O}(G') \tag{5.66}$$

are linear combinations of irreducible gradients on-shell due to eq. (5.65). The rectangular matrix $P^\alpha{}_{\alpha_0}$ has $\text{rank}(P^\alpha{}_{\alpha_0}) = N$. It follows from eqs. (5.12) and (5.65) that there exists an antisymmetric matrix $A^{\alpha\beta}{}_{\alpha_1} = -A^{\beta\alpha}{}_{\alpha_1}(-1)^{\epsilon_\alpha\epsilon_\beta}$ such that

$$X^\alpha{}_{\beta_1} \equiv P^\alpha{}_{\alpha_0}Z^{\alpha_0}{}_{\beta_1} = G'_\beta A^{\beta\alpha}{}_{\beta_1} = \mathcal{O}(G'). \tag{5.67}$$

Because of the rank condition (5.30), one may combine $P^\alpha{}_{\alpha_0}$ and $\omega^{\alpha_1}{}_{\alpha_0}$ into an invertible $N_0 \times N_0$ matrix

$$\left[\begin{array}{c} P \\ \omega \end{array}\right]_{N_0 \times N_0}, \tag{5.68}$$

at least in the vicinity of the constrained surface $G'_\alpha \approx 0$. Next define a rectangular matrix $\bar{P}^{\alpha_0}{}_\alpha$ via

$$\left[\begin{array}{c} P \\ \omega \end{array}\right]_{N_0 \times N_0} [\bar{P}]_{N_0 \times N} = \left[\begin{array}{c} \mathbf{1} \\ 0 \end{array}\right]_{N_0 \times N}, \tag{5.69}$$

and a square matrix $R^{\alpha_0}{}_{\beta_0}$ as

$$[R]_{N_0 \times N_0} \equiv \left[\begin{array}{cc} \bar{P} & Z \end{array}\right]_{N_0 \times N_0}. \tag{5.70}$$

It follows that

$$\left[\begin{array}{c} P \\ \omega \end{array}\right]_{N_0 \times N_0} [R]_{N_0 \times N_0} = \left[\begin{array}{cc} \mathbf{1} & X \\ 0 & \Delta \end{array}\right]_{N_0 \times N_0}, \tag{5.71}$$

cf. (5.29), so that $R^{\alpha_0}{}_{\beta_0}$ must be invertible as well. Next change the coordinates $\lambda^{\alpha_0} = \bar{P}^{\alpha_0}{}_\alpha \lambda'^\alpha + Z^{\alpha_0}{}_{\alpha_1} \lambda'^{\alpha_1}$, or equivalently,

$$[\lambda^{\alpha_0}]_{N_0 \times 1} \;=\; [R]_{N_0 \times N_0} \left[\begin{array}{c} \lambda'^\alpha \\ \lambda'^{\alpha_1} \end{array} \right]_{N_0 \times 1} . \tag{5.72}$$

Therefore one can decompose the "reducible" δ-function

$$
\begin{aligned}
\delta(G_{\omega,\alpha_0}) \;&=\; \int [d\lambda_0] \; e^{\frac{i}{\hbar} G_{\omega,\alpha_0} \lambda^{\alpha_0}} \;=\; \mathrm{sdet}(R^{\alpha_0}{}_{\beta_0}) \int [d\lambda'][d\lambda'_1] \; e^{\frac{i}{\hbar} G'_\alpha \lambda'^\alpha + \pi_{\alpha_1} \Delta^{\alpha_1}{}_{\beta_1} \lambda'^{\beta_1}} \\
&=\; \frac{\mathrm{sdet}(R^{\alpha_0}{}_{\beta_0})}{\mathrm{sdet}(\Delta^{\alpha_1}{}_{\beta_1})} \; \delta(G') \, \delta(\pi_1)
\end{aligned}
\tag{5.73}
$$

in its irreducible components.

There is an almost identical story for the reducible partners F^{α_0}, $\alpha_0 = 1, \ldots, N_0$, in terms of irreducible primed functions F'^α, $\alpha = 1, \ldots, N$, although with some important differences. For instance, we shall assume directly that the reducible *gradients* are linear combinations of irreducible gradients,

$$(F^{\alpha_0} \frac{\overleftarrow{\partial}^r}{\partial \Gamma^A}) \;=\; Q^{\alpha_0}{}_\alpha (F'^\alpha \frac{\overleftarrow{\partial}^r}{\partial \Gamma^A}) . \tag{5.74}$$

The rectangular matrix $Q^{\alpha_0}{}_\alpha$ has rank N. This means there exists a matrix $\bar{Z}^{\alpha_1}{}_{\alpha_0}$ of rank N_1 such that

$$\bar{Z}^{\alpha_1}{}_{\alpha_0} Q^{\alpha_0}{}_\alpha \;=\; 0 . \tag{5.75}$$

Then $\bar{\omega}^{\alpha_0}{}_{\alpha_1}$ is chosen such that

$$\bar{\Delta}^{\alpha_1}{}_{\beta_1} \;\equiv\; \bar{Z}^{\alpha_1}{}_{\alpha_0} \bar{\omega}^{\alpha_0}{}_{\beta_1} \tag{5.76}$$

is invertible, *i.e.*

$$\mathrm{rank}\,(\bar{\Delta}^{\alpha_1}{}_{\beta_1}) \;=\; N_1 . \tag{5.77}$$

One may combine $Q^{\alpha_0}{}_\alpha$ and $\bar{\omega}^{\alpha_0}{}_{\alpha_1}$ into an invertible $N_0 \times N_0$ matrix

$$\left[\begin{array}{cc} Q & \bar{\omega} \end{array} \right]_{N_0 \times N_0} . \tag{5.78}$$

Next define a rectangular matrix $\bar{Q}^\alpha{}_{\alpha_0}$ via

$$[\bar{Q}]_{N \times N_0} \left[\begin{array}{cc} Q & \bar{\omega} \end{array} \right]_{N_0 \times N_0} \;=\; \left[\begin{array}{cc} \mathbf{1} & 0 \end{array} \right]_{N \times N_0} , \tag{5.79}$$

and a square matrix $\bar{R}^{\alpha_0}{}_{\beta_0}$ as

$$[\bar{R}]_{N_0 \times N_0} \;\equiv\; \left[\begin{array}{c} \bar{Q} \\ \bar{Z} \end{array} \right]_{N_0 \times N_0} . \tag{5.80}$$

Finally change the coordinates

$$
\begin{aligned}
[\bar{C}^{\alpha_0}]_{N_0 \times 1} \;&=\; [R]_{N_0 \times N_0} \left[\begin{array}{c} \bar{C}'^\alpha \\ \bar{C}'^{\alpha_1} \end{array} \right]_{N_0 \times 1} , \\
[\bar{C}_{\alpha_0}]_{1 \times N_0} \;&=\; \left[\begin{array}{cc} \bar{C}'_\alpha & \bar{C}'_{\alpha_1} \end{array} \right]_{1 \times N_0} [\bar{R}]_{N_0 \times N_0} , \\
[B^{\alpha_0}]_{N_0 \times 1} \;&=\; [R]_{N_0 \times N_0} \left[\begin{array}{c} B'^\alpha \\ B'^{\alpha_1} \end{array} \right]_{N_0 \times 1} , \\
[\bar{B}_{\alpha_0}]_{1 \times N_0} \;&=\; \left[\begin{array}{cc} \bar{B}'_\alpha & \bar{B}'_{\alpha_1} \end{array} \right]_{1 \times N_0} [\bar{R}]_{N_0 \times N_0} .
\end{aligned}
\tag{5.81}
$$

37

The two superdeterminant actions (5.54) and (5.56) become

$$S_C = (-1)^{\epsilon_A+1}C^A(\frac{\overrightarrow{\partial^l}}{\partial\Gamma^A}G'_\alpha)\bar{C}'^\alpha + (-1)^{\epsilon_{\alpha_1}+1}C_{\alpha_1}\Delta^{\alpha_1}{}_{\beta_1}\bar{C}'^{\beta_1}$$

$$+\bar{C}'_\alpha(F'^\alpha\frac{\overleftarrow{\partial^r}}{\partial\Gamma^A})C^A + \bar{C}'_{\alpha_1}\bar{\Delta}^{\alpha_1}{}_{\beta_1}C^{\beta_1} + \bar{C}'_{\alpha_0}\bar{R}^{\alpha_0}{}_{\beta_0}\tilde{\omega}^{\beta_0\alpha_1}C_{\alpha_1} + \mathcal{O}(G';\pi_1;\pi_1^*) , \quad (5.82)$$

and

$$S_B = \bar{B}'_\alpha(F'^\alpha, G'_\beta)B'^\beta - \bar{B}'_{\alpha_1}\bar{\Delta}^{\alpha_1}{}_{\beta_1}\Delta^{\beta_1}{}_{\gamma_1}B'^{\gamma_1} + \mathcal{O}(G';\pi_1;\pi_1^*) , \quad (5.83)$$

respectively. Integration over $\bar{C}'^{\alpha_1}$ yields a δ-function $\delta(C_{\alpha_1})$. Hence one may drop the $\tilde{\omega}^{\alpha_0\alpha_1}$ term from the S_C action (5.82), so that the Jacobian $J_{\omega,\bar{\omega}}$ factorizes on-shell,

$$J_{\omega,\bar{\omega}} = J'\frac{\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})}{\text{sdet}(R^{\alpha_0}{}_{\beta_0})}\frac{\text{sdet}(\bar{R}^{\alpha_0}{}_{\beta_0})}{\text{sdet}(\bar{\Delta}^{\alpha_1}{}_{\beta_1})} + \mathcal{O}(G';\pi_1;\pi_1^*) , \quad (5.84)$$

with $J' \equiv \text{sdet}(\frac{\partial\bar{\Gamma}'^A}{\partial\Gamma^B})$ and $\bar{\Gamma}'^A \equiv \{F'^\alpha; G'_\alpha\}$. Similarly,

$$\text{sdet}(F^{\alpha_0}_{\bar{\omega}}, G_{\omega,\beta_0})_{\text{ext}} = \text{sdet}(F'^\alpha, G'_\beta)\frac{\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})}{\text{sdet}(R^{\alpha_0}{}_{\beta_0})}\frac{\text{sdet}(\bar{\Delta}^{\alpha_1}{}_{\beta_1})}{\text{sdet}(\bar{R}^{\alpha_0}{}_{\beta_0})} + \mathcal{O}(G';\pi_1;\pi_1^*) . \quad (5.85)$$

Therefore formula (5.52) becomes

$$H = -\ln\sqrt{\frac{J'\ \text{sdet}(F'^\alpha, G'_\beta)}{\rho\ \text{sdet}(R^{\alpha_0}{}_{\beta_0})^2}} + \mathcal{O}(G') , \quad (5.86)$$

and by combining equations (5.37), (5.73) and (5.86), one arrives at the irreducible partition function (3.49) of Corollary 3.5, *i.e.*

$$\mathcal{Z}^{G'}_{[1]} = \int[d\Gamma]\ e^{\frac{i}{\hbar}W}\delta(G')\sqrt{\rho\ J'\ \text{sdet}(F'^\alpha, G'_\beta)} . \quad (5.87)$$

$\square$

In the notation of the above proof one may summarize the bijective correspondence between the constraints $\{G_{\omega,\alpha_0}\} \leftrightarrow \{G'_\alpha; \pi_{\alpha_1}\}$ as follows:

$$[G_{\omega,\alpha_0}]_{1\times N_0} = \begin{bmatrix} G'_\alpha & \pi_{\alpha_1} \end{bmatrix}_{1\times N_0}\begin{bmatrix} P \\ \omega \end{bmatrix}_{N_0\times N_0} , \quad (5.88)$$

$$\begin{bmatrix} G'_\alpha & \pi_{\alpha_1} \end{bmatrix}_{1\times N_0} = [G_{\omega,\alpha_0}]_{1\times N_0}\begin{bmatrix} \bar{P} & Z\Delta^{-1} \end{bmatrix}_{N_0\times N_0} . \quad (5.89)$$

Other useful observations are

$$[\mathbf{1}]_{N_0\times N_0} = \begin{bmatrix} P \\ \omega \end{bmatrix}_{N_0\times N_0}[R]_{N_0\times N_0}\begin{bmatrix} 1 & -X\Delta^{-1} \\ 0 & \Delta^{-1} \end{bmatrix}_{N_0\times N_0}$$

$$= \begin{bmatrix} P \\ \omega \end{bmatrix}_{N_0\times N_0}\begin{bmatrix} \bar{P} & (Z-\bar{P}X)\Delta^{-1} \end{bmatrix}_{N_0\times N_0} , \quad (5.90)$$

which follows from eq. (5.71). Therefore one also has

$$[\mathbf{1}]_{N_0\times N_0} = \begin{bmatrix} \bar{P} & (Z-\bar{P}X)\Delta^{-1} \end{bmatrix}_{N_0\times N_0}\begin{bmatrix} P \\ \omega \end{bmatrix}_{N_0\times N_0}$$

$$= [\bar{P}]_{N_0\times N}[P]_{N\times N_0} + [(Z-\bar{P}X)\Delta^{-1}]_{N_0\times N_1}[\omega]_{N_1\times N_0} , \quad (5.91)$$

and in particular

$$[R^{-1}]_{N_0\times N_0} = \begin{bmatrix} P-X\Delta^{-1}\omega \\ \Delta^{-1}\omega \end{bmatrix}_{N_0\times N_0} . \quad (5.92)$$

5.6 Non-Minimal Approach

With the standard non-minimal Ansatz (5.26) it is mandatory to choose a non-trivial gauge fermion $\Psi \neq 0$. In this Subsection we shall study the most general non-minimal solution

$$
\begin{aligned}
X \;=\; & \mathcal{G}_{A_0}\,\lambda^{A_0} + (i\hbar_{(1)})\,\mathcal{H} + \lambda^*_{A_0}\,\mathcal{Z}^{A_0}{}_{\alpha_1}\,c^{\alpha_1} + \frac{1}{2}\lambda^*_{A_0}\,\mathcal{U}_0^{A_0}{}_{B_0 C_0}\,\lambda^{C_0}\lambda^{B_0}(-1)^{\epsilon_{B_0}+1} \\
& + c^*_{\alpha_1}\,\mathcal{U}_1^{\alpha_1}{}_{\beta_1 A_0}\,\lambda^{A_0} c^{\beta_1}(-1)^{\epsilon_{\beta_1}} + (i\hbar_{(1)})\,\lambda^*_{A_0}\,\mathcal{V}^{A_0}{}_{B_0}\,\lambda^{B_0} + \dots\;,
\end{aligned}
\tag{5.93}
$$

and therefore we may put $\Psi = 0$ without loss of generality. In the above eq. (5.93), which should not be read as a systematic Planck expansion, we have grouped together fields and antifields of the same Planck number, $i.e.$,

$$
\Gamma^{A_{\text{ext}}} \equiv \{\Gamma^A; \pi_{\alpha_1}, \pi^{*\alpha_1}\}\;, \quad \lambda^{A_0} \equiv \{\lambda^{\alpha_0}; -\bar{c}^{*\alpha_1}\}\;, \quad \text{and} \quad \lambda^*_{A_0} \equiv \{\lambda^*_{\alpha_0}; \bar{c}_{\alpha_1}\}\;,
\tag{5.94}
$$

of Planck number 0, 1 and -1, respectively. The minus in front of $\bar{c}^{*\alpha_1}$ in eq. (5.94) is introduced so that $(\lambda^{A_0}, \lambda^*_{B_0})_{[1]} = \delta^{A_0}_{B_0}$. The extended index A_0 runs over $A_0 = 1, \dots, N_0 + N_1$. It is interesting to see how the Quantum Master Equation determines the one-loop correction $\mathcal{H} = \mathcal{H}(\Gamma_{\text{ext}}; \hbar)$ in this general case. The first few consequences of the Quantum Master Equation read

$$
(\mathcal{G}_{A_0}, \mathcal{G}_{B_0})_{\text{ext}} \;=\; \mathcal{G}_{C_0}\,\mathcal{U}_0^{C_0}{}_{A_0 B_0}\;,
\tag{5.95}
$$

$$
\mathcal{G}_{A_0}\mathcal{Z}^{A_0}{}_{\alpha_1} \;=\; 0\;,
\tag{5.96}
$$

$$
(\mathcal{Z}^{A_0}{}_{\beta_1}, \mathcal{G}_{B_0})_{\text{ext}} \;=\; \mathcal{Z}^{A_0}{}_{\alpha_1}\,\mathcal{U}_1^{\alpha_1}{}_{\beta_1 B_0} + (-1)^{(\epsilon_{B_0}+1)(\epsilon_{\beta_1}+1)}\mathcal{U}_0^{A_0}{}_{B_0 C_0}\,\mathcal{Z}^{C_0}{}_{\beta_1} + \mathcal{O}(\mathcal{G})\;,
\tag{5.97}
$$

$$
(\Delta_{\text{ext}}\mathcal{G}_{B_0}) - (\mathcal{H}, \mathcal{G}_{B_0})_{\text{ext}} \;=\; (-1)^{\epsilon_{A_0}}\mathcal{U}_0^{A_0}{}_{A_0 B_0} - (-1)^{\epsilon_{\alpha_1}}\mathcal{U}_1^{\alpha_1}{}_{\alpha_1 B_0} + \mathcal{G}_{A_0}\,\mathcal{V}^{A_0}{}_{B_0}\;.
\tag{5.98}
$$

The extended constraints $\mathcal{G}_{A_0} = \mathcal{G}_{A_0}(\Gamma_{\text{ext}}; \hbar)$ and the extended generators $\mathcal{Z}^{A_0}{}_{\beta_1} = \mathcal{Z}^{A_0}{}_{\beta_1}(\Gamma_{\text{ext}}; \hbar)$ are both reducible sets of functions,

$$
\mathcal{G}_{A_0} \equiv \{\mathcal{G}_{\alpha_0}; \mathcal{G}_{\alpha_1}\}\;, \qquad \mathcal{Z}^{A_0}{}_{\beta_1} \equiv \{\mathcal{Z}^{\alpha_0}{}_{\beta_1}; \Delta^{\alpha_1}{}_{\beta_1}\}\;.
\tag{5.99}
$$

Here $\Delta^{\alpha_1}{}_{\beta_1}$ is defined as the matrix from the quadratic $\{\bar{c}_{\alpha_1}; c^{\beta_1}\}$ term in the action (5.93). A first-stage theory has a total of $4(N_0 + N_1)$ fields and antifields, and we know that the Hessian for X has half rank on stationary field configurations. Putting all the antifields to zero, the Hessian must have full rank $(= 2(N_0 + N_1))$ in the field-field quadrant. Hence it follows that $\mathcal{G}_{\alpha_0}$ is irreducible and $\Delta^{\alpha_1}{}_{\beta_1}$ has maximal rank. Equations (5.95) and (5.96) therefore show that

$$
\mathcal{G}_{\alpha_1} \;=\; \mathcal{O}(\mathcal{G}_{\alpha_0})\;, \qquad\qquad \mathcal{Z}^{\alpha_0}{}_{\beta_1} \;=\; \mathcal{O}(\mathcal{G}_{\alpha_0})\;,
\tag{5.100}
$$

respectively. We arrive at the following version of Lemma 5.1:

Lemma 5.7 : *The path integrations over the Faddeev-Popov ghost pair $\{c^{\alpha_1}; \bar{c}_{\alpha_1}\}$ can be performed explicitly. The first-level partition function (5.31) thereby simplifies to*

$$
\mathcal{Z}_{[1]} \;=\; \int [d\Gamma][d\lambda_0][d\pi_1]\,\rho\,e^{\frac{i}{\hbar}W + \frac{i}{\hbar}\mathcal{G}_{\alpha_0}\lambda^{\alpha_0} - \mathcal{H}}\,\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})\;.
\tag{5.101}
$$

One can also prove a version of Theorem 3.2. For that purpose, let $\bar{\Gamma}^{A_{\text{ext}}} \equiv \{\mathcal{F}^{\alpha_0}; \mathcal{G}_{\alpha_0}\}$ and $\mathcal{J} \equiv \text{sdet}(\frac{\partial \bar{\Gamma}^{A_{\text{ext}}}}{\partial \Gamma^{B_{\text{ext}}}})$.

Theorem 5.8 : *The Quantum Master Equation for the non-minimal X in eq. (5.93) implies that the quantum correction $\mathcal{H}$ is given by the following square root formula*

$$
\mathcal{H} \;=\; -\ln\sqrt{\frac{\mathcal{J}\,\text{sdet}(\mathcal{F}^{\alpha_0}, \mathcal{G}_{\beta_0})_{\text{ext}}}{\rho\,\text{sdet}(\Delta^{\alpha_1}{}_{\beta_1})^2}} + \mathcal{O}(\mathcal{G})\;.
\tag{5.102}
$$

Table 1: First-level fields in the general reducible case. Antifields are not shown. The last field in each row is a minimal field.

Grassm.	$\epsilon_{a_5}+1$	ϵ_{a_4}	$\epsilon_{a_3}+1$	ϵ_{a_2}	$\epsilon_{a_1}+1$	ϵ_{a_π}	$\epsilon_{a_{ext}}$	$\epsilon_{a_1}+1$	ϵ_{a_2}	$\epsilon_{a_3}+1$	ϵ_{a_4}	$\epsilon_{a_5}+1$
Pl	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
	$\bar{c}_{a_5}$	$\bar{c}_{a_4}$	$\bar{c}_{a_3}$	$\bar{c}_{a_2}$	$\bar{c}_{a_1}$	π_{a_π}	$\lambda^{a_{ext}}$	c^{a_1}	c^{a_2}	c^{a_3}	c^{a_4}	c^{a_5}
0							λ^{α_0}					
1					$\bar{c}_{1,\alpha_1}$	π_{α_1}		$c_1^{\alpha_1}$				
2				$\bar{c}_{2,\alpha_2}$	$\bar{c}_{1,\alpha_2}$		λ^{α_2}	$c_1^{\alpha_2}$	$c_2^{\alpha_2}$			
3			$\bar{c}_{3,\alpha_3}$	$\bar{c}_{2,\alpha_3}$	$\bar{c}_{1,\alpha_3}$	π_{α_3}		$c_1^{\alpha_3}$	$c_2^{\alpha_3}$	$c_3^{\alpha_3}$		
4		$\bar{c}_{4,\alpha_4}$	$\bar{c}_{3,\alpha_4}$	$\bar{c}_{2,\alpha_4}$	$\bar{c}_{1,\alpha_4}$		λ^{α_4}	$c_1^{\alpha_4}$	$c_2^{\alpha_4}$	$c_3^{\alpha_4}$	$c_4^{\alpha_4}$	
5	$\bar{c}_{5,\alpha_5}$	$\bar{c}_{4,\alpha_5}$	$\bar{c}_{3,\alpha_5}$	$\bar{c}_{2,\alpha_5}$	$\bar{c}_{1,\alpha_5}$	π_{α_5}		$c_1^{\alpha_5}$	$c_2^{\alpha_5}$	$c_3^{\alpha_5}$	$c_4^{\alpha_5}$	$c_5^{\alpha_5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Stage	Antighosts					Lagr. Mult.		Ghosts				

PROOF OF THEOREM 5.8: The eq. (5.97) can be rewritten as

$$(\ln \mathrm{sdet}(\Delta^{\alpha_1}{}_{\beta_1}),\ \mathcal{G}_{B_0})_{\text{ext}} \ = \ (-1)^{\epsilon_{\alpha_1}}\mathcal{U}_1^{\alpha_1}{}_{\alpha_1 B_0} - (-1)^{\epsilon_{\alpha_1}}\mathcal{U}_0^{\alpha_1}{}_{\alpha_1 B_0} + \mathcal{O}(\mathcal{G})\ , \tag{5.103}$$

which with the help of eq. (5.98) becomes

$$(\Delta_{\text{ext}}\mathcal{G}_{B_0}) - (\mathcal{H} - \ln \mathrm{sdet}(\Delta^{\alpha_1}{}_{\beta_1}),\ \mathcal{G}_{B_0})_{\text{ext}} \ = \ (-1)^{\epsilon_{\alpha_0}}\mathcal{U}_0^{\alpha_0}{}_{\alpha_0 B_0} + \mathcal{O}(\mathcal{G})\ . \tag{5.104}$$

Next one proceed as in the proof of Theorem 3.2.

$\square$

Equations (5.101) and (5.102) leads to eq. (5.64) in Corollary 5.4. So the non-minimal approach agrees with the previous approach of Subsection 5.4.

5.7 Higher-Stage Reducibility and Second-Class Constraints

Adapting the reducible zeroth-level recipe of Ref. [2] to a first-level theory of stage s, one introduces $2\sum_{i=0}^{s}(2i+1)N_i$ first-level fields and antifields as indicated in Table 1. It is useful to group together fields with the same Planck number assignment. Hence we write

$$\lambda^{a_{\text{ext}}} \ \equiv \ \{\lambda^{\alpha_0}; \lambda^{\alpha_2}; \lambda^{\alpha_4}; \ldots\}\ , \tag{5.105}$$

$$\pi_{a_\pi} \ \equiv \ \{\pi_{\alpha_1}; \pi_{\alpha_3}; \pi_{\alpha_5}; \ldots\}\ , \tag{5.106}$$

$$c^{a_i} \ \equiv \ \{c_i^{\alpha_i}; c_i^{\alpha_{i+1}}; c_i^{\alpha_{i+2}}; \ldots; c_i^{\alpha_s}\}\ , \qquad\qquad i = 1, \ldots, s\ , \tag{5.107}$$

$$\bar{c}_{a_i} \ \equiv \ \{\bar{c}_{i,\alpha_i}; \bar{c}_{i,\alpha_{i+1}}; \bar{c}_{i,\alpha_{i+2}}; \ldots; \bar{c}_{i,\alpha_s}\}\ , \qquad\qquad i = 1, \ldots, s\ . \tag{5.108}$$

The total space is then

$$\Gamma_{[1]}^A \ \equiv \ \left\{\Gamma^{A_{\text{ext}}}; \lambda^{a_{\text{ext}}}, \lambda^*_{a_{\text{ext}}}; c^{a_1}, c^*_{a_1}, \bar{c}_{a_1}, \bar{c}^{*a_1}; c^{a_2}, c^*_{a_2}, \bar{c}_{a_2}, \bar{c}^{*a_2}; \ldots; c^{a_s}, c^*_{a_s}, \bar{c}_{a_s}, \bar{c}^{*a_s}\right\}\ , \tag{5.109}$$

$$\Gamma^{A_{\text{ext}}} \ \equiv \ \left\{\Gamma^A; \pi_{a_\pi}, \pi^{*a_\pi}\right\}\ . \tag{5.110}$$

40

Here the indices run

$$
\begin{aligned}
A_{\text{ext}} &= 1,\ldots,2M_{\text{odd}} \quad, & a_\pi &= 1,\ldots,M_{\text{odd}}-N \quad, \\
a_{\text{ext}} &= 1,\ldots,M_{\text{even}} \quad, & a_i &= 1,\ldots,M_s-M_{i-1} \quad,
\end{aligned}
\tag{5.111}
$$

where

$$
M_{\text{odd}} \;\equiv\; \sum_{\substack{i=-1,\ldots,s\\ i\ \text{odd}}} N_i \quad, \quad
M_{\text{even}} \;\equiv\; \sum_{\substack{i=0,\ldots,s\\ i\ \text{even}}} N_i \quad, \quad
M_{\text{odd}} \;=\; M_{\text{even}}+N_D \quad,
\tag{5.112}
$$

$$
M_i \;\equiv\; \sum_{j=-1}^{i} N_j \qquad, \qquad N_{-1} \;\equiv\; N \;.
$$

The Planck-number operator is

$$
\mathsf{PI} \;=\; -\left(\lambda^{a_{\text{ext}}}\lambda^*_{a_{\text{ext}}} + \sum_{i=1}^{s}[(i+1)c^*_{a_i}c^{a_i} - i\bar{c}_{a_i}\bar{c}^{*a_i}], \;\cdot\;\right)_{[1]} + \hbar_{(1)}\frac{\partial}{\partial\hbar_{(1)}} \;,
\tag{5.113}
$$

or equivalently,

$$
\begin{aligned}
\mathsf{PI}(\lambda^{a_{\text{ext}}}) &= 1 &,\quad \mathsf{PI}(c^{a_i}) &= i+1 &,\quad \mathsf{PI}(\bar{c}_{a_i}) &= -i \;, \\
\mathsf{PI}(\lambda^*_{a_{\text{ext}}}) &= -1 &,\quad \mathsf{PI}(c^*_{a_i}) &= -(i+1) &,\quad \mathsf{PI}(\bar{c}^{*a_i}) &= i \;, \\
\mathsf{PI}(\Gamma^{A_{\text{ext}}}) &= 0 &,\quad \mathsf{PI}(\hbar_{(1)}) &= 1 &,\quad \mathsf{PI}(\hbar) &= 0 \;.
\end{aligned}
\tag{5.114}
$$

The standard Ansatz for the non-minimal action is

$$
X_D \;=\; X_{D,\text{min}}(\Gamma^A_{[1]\,\text{min}};\hbar_{[1]}) + \sum_{\substack{j=1,\ldots,s\\ j\ \text{odd}}} \pi_{\alpha_j}\,\bar{c}_1^{*\alpha_j} + \sum_{\substack{i,j=2,\ldots,s\\ j-i\ \text{even},\geq 0}} \bar{c}_{i-1,\alpha_j}\,\bar{c}_i^{*\alpha_j}
$$

$$
+\; \sum_{\substack{j=2,\ldots,s\\ j\ \text{even}}} \lambda^*_{\alpha_j}\,c_1^{\alpha_j} + \sum_{\substack{i,j=2,\ldots,s\\ j-i\ \text{odd},>0}} c^*_{i-1,\alpha_j}\,c_i^{\alpha_j} \;.
\tag{5.115}
$$

A choice of the gauge fermion reads

$$
-\Psi \;=\; \bar{c}_{a_1}\,\omega_1^{a_1}{}_{a_{\text{ext}}}\,\lambda^{a_{\text{ext}}} + \sum_{i=2}^{s} \bar{c}_{a_i}\,\omega_i^{a_i}{}_{a_{i-1}}\,c^{a_{i-1}} \;, \qquad\qquad \mathsf{PI}(\Psi) \;=\; 0 \;,
\tag{5.116}
$$

where a simple choice for the matrices $\omega_i^{a_i}{}_{a_{i-1}}$, $i=1,\ldots,s$, is indicated in Table 2. For notational reasons it is convenient to trivially extend the matrix $\omega_1^{a_1}{}_{a_{\text{ext}}} \to \omega_1^{a_1}{}_{a_0}$ with zero columns such that the column index reads $a_0 \equiv \{\alpha_0;\alpha_1;\alpha_2;\alpha_3;\ldots\}$ rather than $a_{\text{ext}} \equiv \{\alpha_0;\alpha_2;\alpha_4;\alpha_6;\ldots\}$. The first-level partition function is given by

$$
\mathcal{Z}^{\Psi}_{[1]D} \;=\; \int d\mu_D\, e^{\frac{i}{\hbar}(W_D+X_D^{\Psi})}\Big|_{\lambda^*,c^*,\bar{c}^*,\pi^*=0}\,\delta(\Theta^a) \;=\; \int d\mu_D\, e^{\frac{i}{\hbar}(W_D^{\Psi}+X_D)}\Big|_{\Sigma}\,\delta(\Theta^a) \;,
\tag{5.117}
$$

with a measure

$$
d\mu_D \;=\; \rho_D[d\Gamma][d\lambda][dc][d\bar{c}][d\pi] \;,
\tag{5.118}
$$

Table 2: The simplest choice of a rectangular gauge-fixing matrix $\omega_i^{a_i}{}_{b_{i-1}}$, $i = 1, \ldots, s$.

$a_i\backslash b_{i-1}$	β_{i-1}	β_i	β_{i+1}	β_{i+2}	β_{i+3}	β_{i+4}	$\cdots$	β_s
α_i	$\omega_i^{\alpha_i}{}_{\beta_{i-1}}$		$\omega_i^{\alpha_i}{}_{\beta_{i+1}}$					
α_{i+1}								
α_{i+2}			$\omega_i^{\alpha_{i+2}}{}_{\beta_{i+1}}$		$\omega_i^{\alpha_{i+2}}{}_{\beta_{i+3}}$			
α_{i+3}								
α_{i+4}					$\omega_i^{\alpha_{i+4}}{}_{\beta_{i+3}}$	$\ddots$		
$\vdots$								
α_s						$\ddots$		

and a gauge-fixing surface Σ specified by

$$\lambda^*_{a_{\text{ext}}} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial\lambda^{a_{\text{ext}}}} \;\; = \;\; \bar{c}_{a_1} E(\text{ad}_D\Psi)\omega_1^{a_1}{}_{a_{\text{ext}}} \; , \tag{5.119}$$

$$c^*_{a_i} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial c^{a_i}} \;\; = \;\; \bar{c}_{a_{i+1}} E(\text{ad}_D\Psi)\omega_{i+1}^{a_{i+1}}{}_{a_i} \; , \qquad\qquad i = 1, \ldots, s-1 \; , \tag{5.120}$$

$$c^*_{a_s} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial c^{a_s}} \;\; = \;\; 0 \; , \tag{5.121}$$

$$\bar{c}^{*a_1} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial\bar{c}_{a_1}} \;\; = \;\; E(\text{ad}_D\Psi)\omega_1^{a_1}{}_{a_{\text{ext}}} \; \lambda^{a_{\text{ext}}} \; , \tag{5.122}$$

$$\bar{c}^{*a_i} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial\bar{c}_{a_i}} \;\; = \;\; E(\text{ad}_D\Psi)\omega_i^{a_i}{}_{a_{i-1}} \; c^{a_{i-1}} \; , \qquad\qquad i = 2, \ldots, s \; , \tag{5.123}$$

$$\pi^{*a_\pi} \;\; = \;\; -E(\text{ad}_D\Psi)\frac{\partial\Psi}{\partial\pi_{a_\pi}} \;\; = \;\; 0 \; , \tag{5.124}$$

cf. the prescription (3.28).

Similar to the first-stage case the generators $Z_i^{\alpha_{i-1}}{}_{\beta_i}$ and the gauge-fixing matrices $\omega_i^{a_i}{}_{b_{i-1}}$ give rise to Faddeev-Popov matrices $\Delta_i^{a_i}{}_{b_i}$ as indicated in Table 3. The pertinent rank conditions to the gauge-fixing matrices $\omega_i^{a_i}{}_{b_{i-1}}$ are such that the sequence of matrices $\Delta^{a_i}{}_{b_i}$, $i = 1, \ldots, s$, becomes invertible. Explicit rank conditions for the $\omega_i^{a_i}{}_{b_{i-1}}$ matrices for theories of stage 1, 2 and 3 can be found in the original paper [2].

Lemma 5.9 : *The path integrations over the ghost pairs* $\{c^{a_1}; \bar{c}_{a_1}; \ldots; c^{a_s}; \bar{c}_{a_s}\}$ *can be performed explicitly. The first-level partition function thereby simplifies to*

$$\mathcal{Z}_{D[1]} \;\; = \;\; \int[d\Gamma][d\lambda][d\pi] \; \rho_D \; e^{\frac{i}{\hbar}W_D + \frac{i}{\hbar}G_{\omega,a_{\text{ext}}}\lambda^{a_{\text{ext}}} - H} \; \delta(\Theta^a) \; \prod_{i=1}^{s} \text{sdet}(\Delta^{a_i}{}_{b_i})^{-(-1)^i} \; , \tag{5.125}$$

where $G_{\omega,a_{\text{ext}}}$ *are the* ω-*deformed gauge-fixing constraints,*

$$G_{\omega,a_{\text{ext}}} \;\; \equiv \;\; \{G_{\omega,\alpha_0}; G_{\omega,\alpha_2}; G_{\omega,\alpha_4}; G_{\omega,\alpha_6}; \ldots\} \; , \tag{5.126}$$

$$G_{\omega,\alpha_0} \;\; \equiv \;\; G_{\alpha_0} + \pi_{\alpha_1} \, \omega_1^{\alpha_1}{}_{\alpha_0} \; , \tag{5.127}$$

$$G_{\omega,\alpha_i} \;\; \equiv \;\; \pi_{\alpha_{i+1}} \, \omega_1^{\alpha_{i+1}}{}_{\alpha_i} \; , \qquad\qquad i = 2, \ldots, s \; , \; i \text{ even} \; , \tag{5.128}$$

Table 3: A square $\Delta^{a_i}_{i\ b_i}$ matrix corresponding to the gauge-fixing matrices $\omega^{a_i}_{i\ b_{i-1}}$ of Table 2. The entry corresponding to the first row and the first column is $\Delta^{\alpha_i}_{i\ \beta_i} \equiv \omega^{\alpha_i}_{i\ \alpha_{i-1}} Z^{\alpha_{i-1}}_{i\ \beta_i}$.

$a_i\backslash b_i$	β_i	β_{i+1}	β_{i+2}	β_{i+3}	β_{i+4}	$\cdots$	β_s
α_i	$\Delta^{\alpha_i}_{i\ \beta_i}$	$\omega^{\alpha_i}_{i\ \beta_{i+1}}$					
α_{i+1}	$\omega^{\alpha_{i+1}}_{i+1\ \beta_i}$		$\omega^{\alpha_{i+1}}_{i+1\ \beta_{i+2}}$				
α_{i+2}		$\omega^{\alpha_{i+2}}_{i\ \beta_{i+1}}$		$\omega^{\alpha_{i+2}}_{i\ \beta_{i+3}}$			
α_{i+3}			$\omega^{\alpha_{i+3}}_{i+1\ \beta_{i+2}}$		$\omega^{\alpha_{i+3}}_{i+1\ \beta_{i+4}}$		
α_{i+4}				$\omega^{\alpha_{i+4}}_{i\ \beta_{i+3}}$			
$\vdots$							$\ddots$
α_s					$\ddots$		

and G_{α_0} *are the original reducible constraints.*

SKETCHED PROOF OF LEMMA 5.9: The gauge-fixed action

$$S \;=\; (\frac{\hbar_{(1)}}{\hbar} W^{\Psi}_{D} + X_D)|_{\Sigma} \;=\; S_0 + S_{FP} + V \;, \tag{5.129}$$

splits into a "constant" part

$$S_0 \;\equiv\; \frac{\hbar_{(1)}}{\hbar} W_D + G_{\omega,a_{\text{ext}}} \lambda^{a_{\text{ext}}} + (i\hbar_{(1)})H \tag{5.130}$$

that is independent of the ghosts and antighosts $\{c^{a_i}; \bar{c}_{a_i}\}$, a Faddeev-Popov part

$$S_{FP} \;\equiv\; \sum_{i=1}^{s} \bar{c}_{a_i} \, \Delta^{a_i}_{i\ b_i} \, c^{b_i} \tag{5.131}$$

that is quadratic in $\{c^{a_i}; \bar{c}_{a_i}\}$, and an part V that contains all interaction terms, tadpole terms and terms quadratic in the antighosts $\bar{c}_{a_i}$. The multiplicity condition that generalizes the rule (5.43) states, that for each term in V there exists an $i = 1, \ldots, s$, such that there are fewer ghosts c^{a_i} than antighosts $\bar{c}_{a_i}$.

$\square$

Next, let $J_{D,\omega,\bar{\omega}} \equiv \mathrm{sdet}(\frac{\partial \bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}}}{\partial \Gamma^{B}_{\text{ext}}})$, $\bar{\Gamma}^{A_{\text{ext}}}_{\omega,\bar{\omega}} \equiv \{F^{a_{\text{ext}}}_{\bar{\omega}}; G_{\omega,a_{\text{ext}}}; \Theta^a\}$ and $\Gamma^{A_{\text{ext}}} \equiv \{\Gamma^A; \pi_{a_\pi}, \pi^{*a_\pi}\}$. Then

Theorem 5.10 : *The Quantum Master Equation for* $X_{D,\min}$ *implies that the quantum correction* H *is given by the following square root formula*

$$H \;=\; -\ln\left[\left.\sqrt{\frac{J_{D,\omega,\bar{\omega}}\ \mathrm{sdet}(F^{a_{\text{ext}}}_{\bar{\omega}}, G_{\omega,b_{\text{ext}}})_{D,\text{ext}}}{\rho_D}}\right|_{\pi,\pi^*=0} \prod_{i=1}^{s} \mathrm{sdet}(\Delta^{a_i}_{\ b_i})^{(-1)^i} \right] + \mathcal{O}(G_{\alpha_0}; \Theta) \;. \tag{5.132}$$

43

In addition, let $\mathcal{J}_D \equiv \mathrm{sdet}(\frac{\partial \bar{\Gamma}^{A_{\mathrm{ext}}}}{\partial \Gamma^{B_{\mathrm{ext}}}})$ and $\bar{\Gamma}^{A_{\mathrm{ext}}} \equiv \{\mathcal{F}^{a_{\mathrm{ext}}}; \mathcal{G}_{a_{\mathrm{ext}}}; \Theta^a\}$. Then

Corollary 5.11 : *The partition function of stage s*

$$\mathcal{Z}^{\mathcal{G}}_{D[1]} = \int [d\Gamma_{\mathrm{ext}}] \, e^{\frac{i}{\hbar}W_D} \, \delta(\pi^{*a_\pi}) \, \delta(\Theta^a) \, \delta(\mathcal{G}_{a_{\mathrm{ext}}}) \sqrt{\rho_D \, \mathcal{J}_D \, \mathrm{sdet}(\mathcal{F}^{a_{\mathrm{ext}}}, \mathcal{G}_{b_{\mathrm{ext}}})_{D,\mathrm{ext}}} \tag{5.133}$$

is independent of $\mathcal{G}_{a_{\mathrm{ext}}}$'s that are in involution with respect to the $(\cdot, \cdot)_{D,\mathrm{ext}}$-bracket.

Note that the alternating product of superdeterminants [16] has cancelled out of the final expression. A proof of Theorem 5.10 will appear elsewhere.

5.8   Non-Minimal Approach for Higher-Stages

We consider the most general non-minimal solution

$$\begin{aligned}
X_D &= \mathcal{G}_{A_0} \lambda^{A_0} + (i\hbar_{(1)}) \, \mathcal{H} + \lambda^*_{A_0} \, \mathcal{Z}^{A_0}{}_{A_1} \, c^{A_1} + \sum_{i=1}^{s-1} c^*_{A_i} \, \mathcal{Z}^{A_i}{}_{A_{i+1}} \, c^{A_{i+1}} \\
&\quad + \frac{1}{2} \lambda^*_{A_0} \, \mathcal{U}^{A_0}_{0\,B_0 C_0} \, \lambda^{C_0} \lambda^{B_0} (-1)^{\epsilon_{B_0}+1} + \sum_{i=1}^{s} c^*_{A_i} \, \mathcal{U}^{A_1}_{i\,B_i C_0} \, \lambda^{C_0} c^{B_i} (-1)^{\epsilon_{B_i}+i+1} \\
&\quad + (i\hbar_{(1)}) \, \lambda^*_{A_0} \, \mathcal{V}^{A_0}{}_{B_0} \, \lambda^{B_0} + \dots \; .
\end{aligned} \tag{5.134}$$

In the above eq. (5.134) we have grouped together fields of the same Planck number, *i.e.* ,

$$\lambda^{A_0} \equiv \begin{bmatrix} \lambda^{a_{\mathrm{ext}}} \\ -\bar{c}^{*a_1} \end{bmatrix} \quad , \quad c^{A_i} \equiv \begin{bmatrix} c^{a_i} \\ -\bar{c}^{*a_{i+1}} \end{bmatrix} \quad , \quad c^{A_s} \equiv [c^{a_s}] \; ,$$

$$\lambda^*_{A_0} \equiv \begin{bmatrix} \lambda^*_{a_{\mathrm{ext}}} \\ \bar{c}_{a_1} \end{bmatrix} \quad , \quad c^*_{A_i} \equiv \begin{bmatrix} c^*_{a_i} \\ \bar{c}_{a_{i+1}} \end{bmatrix} \quad , \quad c^*_{A_s} \equiv [c^*_{a_s}] \; , \tag{5.135}$$

$$i = 1, \dots, s-1 \quad .$$

The first few consequences of the Quantum Master Equation read

$$(\mathcal{G}_{A_0}, \mathcal{G}_{B_0})_{D,\mathrm{ext}} = \mathcal{G}_{C_0} \, \mathcal{U}^{C_0}_{0\,A_0 B_0} \; , \tag{5.136}$$

$$\mathcal{G}_{A_0} \mathcal{Z}^{A_0}{}_{A_1} = 0 \; , \tag{5.137}$$

$$\mathcal{Z}^{A_{i-1}}{}_{A_i} \mathcal{Z}^{A_i}{}_{A_{i+1}} = \mathcal{O}(\mathcal{G}) \; , \tag{5.138}$$

$$\begin{aligned}
(\mathcal{Z}^{A_i}{}_{B_{i+1}}, \mathcal{G}_{B_0})_{D,\mathrm{ext}} &= \mathcal{Z}^{A_i}{}_{A_{i+1}} \, \mathcal{U}^{A_{i+1}}_{i+1\,B_{i+1} B_0} \\
&\quad - (-1)^{(\epsilon_{B_0}+1)(\epsilon_{C_i}+\epsilon_{B_{i+1}})} \mathcal{U}^{A_i}_{i\,C_i B_0} \, \mathcal{Z}^{C_i}{}_{B_{i+1}} + \mathcal{O}(\mathcal{G}) \; , \tag{5.139}
\end{aligned}$$

$$(\Delta_{D,\mathrm{ext}} \mathcal{G}_{B_0}) - (\mathcal{H}, \mathcal{G}_{B_0})_{D,\mathrm{ext}} = \sum_{i=0}^{s} (-1)^{\epsilon_{A_i}+i} \mathcal{U}^{A_i}_{i\,A_i B_0} + \mathcal{G}_{A_0} \, \mathcal{V}^{A_0}{}_{B_0} \; . \tag{5.140}$$

Ignoring at first the second-class constraints, the Hessian for X has half rank on stationary field configurations. Putting all the antifields to zero, the Hessian must have full rank in the field-field quadrant. It follows that the other three quadrants of the Hessian must vanish on stationary field configurations. In the presence of second-class constraints the same reasoning can be used in the physical subsector. We conclude that the extended constraints $\mathcal{G}_{A_0} = \mathcal{G}_{A_0}(\Gamma_{\mathrm{ext}}; \hbar)$ and the extended generators $\mathcal{Z}^{A_{i-1}}{}_{B_i} = \mathcal{Z}^{A_{i-1}}_i{}_{B_i}(\Gamma_{\mathrm{ext}}; \hbar)$ are both reducible sets of functions,

$$\begin{aligned}
\mathcal{G}_{A_0} &\equiv \begin{bmatrix} \mathcal{G}_{a_{\mathrm{ext}}} & \mathcal{G}_{a_1} \end{bmatrix} \quad , \quad \mathcal{Z}^{A_{i-1}}{}_{B_i} = \begin{bmatrix} \mathcal{O}(\mathcal{G};\Theta) & \mathcal{O}(\mathcal{G};\Theta) \\ \Delta^{a_i}{}_{b_i} & \mathcal{O}(\mathcal{G};\Theta) \end{bmatrix} \quad , \quad \mathcal{Z}^{A_{s-1}}{}_{B_s} = \begin{bmatrix} \mathcal{O}(\mathcal{G};\Theta) \\ \Delta^{a_s}{}_{b_s} \end{bmatrix} \; , \\
\mathcal{G}_{a_1} &= \mathcal{O}(\mathcal{G}_{a_{\mathrm{ext}}}; \Theta) \quad , \quad\quad\quad i = 1, \dots, s-1 \quad .
\end{aligned}$$
$$\tag{5.141}$$

Here $\Delta^{a_i}{}_{b_i}$ is defined as the matrix from the quadratic $\{\bar{c}_{a_i}; c^{b_i}\}$ term in the action (5.134). Hence it follows that $\mathcal{G}_{a_{\text{ext}}}$ is irreducible and $\Delta^{a_i}{}_{b_i}$, $i = 1, \ldots, s$, have maximal rank. We arrive at the following version of Lemma 5.9:

Lemma 5.12 : *The path integrations over the ghost pairs $\{c^{a_1}; \bar{c}_{a_1}; \ldots; c^{a_s}; \bar{c}_{a_s}\}$ can be performed explicitly. The first-level partition function thereby simplifies to*

$$\mathcal{Z}_{D[1]} \;=\; \int [d\Gamma][d\lambda][d\pi] \; \rho_D \; e^{\frac{i}{\hbar}W_D + \frac{i}{\hbar}\mathcal{G}_{a_{\text{ext}}}\lambda^{a_{\text{ext}}} - \mathcal{H}} \; \delta(\Theta^a) \; \prod_{i=1}^{s} \operatorname{sdet}(\Delta^{a_i}{}_{b_i})^{-(-1)^i} \;. \tag{5.142}$$

One can also prove a version of Theorem 5.10. For that purpose, let $\mathcal{J}_D \equiv \operatorname{sdet}(\frac{\partial \bar{\Gamma}^{A_{\text{ext}}}}{\partial \Gamma^{B_{\text{ext}}}})$, $\bar{\Gamma}^{A_{\text{ext}}} \equiv \{\mathcal{F}^{a_{\text{ext}}}; \mathcal{G}_{a_{\text{ext}}}; \Theta^a\}$ and $\Gamma^{A_{\text{ext}}} \equiv \{\Gamma^A; \pi_{a_\pi}, \pi^{*a_\pi}\}$. Then

Theorem 5.13 : *The Quantum Master Equation for the non-minimal X_D in eq. (5.134) implies that the quantum correction $\mathcal{H} = \mathcal{H}(\Gamma_{\text{ext}}; \hbar)$ is given by the following square root formula*

$$\mathcal{H} \;=\; -\ln \left[\sqrt{\frac{\mathcal{J}_D \; \operatorname{sdet}(\mathcal{F}^{a_{\text{ext}}}, \mathcal{G}_{b_{\text{ext}}})_{D,\text{ext}}}{\rho_D}} \prod_{i=1}^{s} \operatorname{sdet}(\Delta^{a_i}{}_{b_i})^{(-1)^i} \right] + \mathcal{O}(\mathcal{G}; \Theta) \;. \tag{5.143}$$

PROOF OF THEOREM 5.13:   Eq. (5.139) can be rewritten as

$$(\ln \operatorname{sdet}(\Delta^{a_i}{}_{b_i}), \; \mathcal{G}_{B_0})_{D,\text{ext}} \;=\; (-1)^{\epsilon_{a_i}} \mathcal{U}_i^{a_i}{}_{a_i B_0} - (-1)^{\epsilon_{a_i}} \mathcal{U}_{i-1}^{a_i}{}_{a_i B_0} + \mathcal{O}(\mathcal{G}) \;, \tag{5.144}$$

which with the help of eq. (5.140) becomes

$$(\Delta_{D,\text{ext}} \mathcal{G}_{B_0}) - \left(\mathcal{H} + \sum_{i=1}^{s} (-1)^i \ln \operatorname{sdet}(\Delta^{a_i}{}_{b_i}), \; \mathcal{G}_{B_0} \right)_{D,\text{ext}} \;=\; (-1)^{\epsilon_{a_{\text{ext}}}} \mathcal{U}_0^{a_{\text{ext}}}{}_{a_{\text{ext}} B_0} + \mathcal{O}(\mathcal{G}) \;. \tag{5.145}$$

Next one proceed as in the proof of Theorem 3.2.

$\square$

Equations (5.142) and (5.143) leads to eq. (5.133) in Corollary 5.11. So the non-minimal approach agrees with the approach of Subsection 5.7.

6   Higher Level Formalism

6.1   Recursive Construction

In the irreducible $n'th$-$level$ formalism one introduces N Lagrange multipliers $\lambda^\alpha_{(n)}$ of Grassmann parity $\epsilon^{(n)}_\alpha \equiv \epsilon^{(0)}_\alpha + n$ and N antifields $\lambda^*_{(n)\alpha}$, which we collectively call the $n'th$-$level$ $fields$. The phase space variables

$$\Gamma^A_{[n]} \;\equiv\; \left\{ \Gamma^A_{[n-1]}; \lambda^\alpha_{(n)}, \lambda^*_{(n)\alpha} \right\} \tag{6.1}$$

for the $n'th$-level formalism thus consists of fields of levels $\leq n$. The idea is roughly that the $n'th$-level Lagrange multipliers λ^α should gauge-fix the $(n-1)'th$-level first-level antifields $\lambda^*_{(n-1)\alpha}$, although this is just one gauge-fixing choice out of infinitely many.

In the $n'th$-level formalism, one first lifts the previous $(n-1)'th$-level phase space $\Gamma^A_{[n-1]}$ to a fully covariant status, i.e. one allows for general coordinate transformations $\Gamma^A_{[n-1]} \to \Gamma'^A_{[n-1]} = \Gamma'^A_{[n-1]}(\Gamma_{[n-1]}; \hbar_{[n-1]})$ that preserve the Planck number symmetries of the previous levels, so that the $(n-1)'th$-level odd Laplacian becomes of the covariant form

$$\Delta_{[n-1]} = \frac{(-1)^{\epsilon_A}}{2\rho_{[n-1]}} \frac{\overrightarrow{\partial^l}}{\partial\Gamma^A_{[n-1]}} \rho_{[n-1]} E^{AB}_{[n-1]} \frac{\overrightarrow{\partial^l}}{\partial\Gamma^B_{[n-1]}} \tag{6.2}$$

with a symplectic metric $E^{AB}_{[n-1]} = E^{AB}_{[n-1]}(\Gamma_{[n-1]}; \hbar_{[n-1]})$ and a measure density $\rho_{[n-1]} = \rho_{[n-1]}(\Gamma_{[n-1]}; \hbar_{[n-1]})$. Secondly, one defines a $n'th$-level odd Laplacian

$$\Delta_{[n]} \equiv \Delta_{[n-1]} + (-1)^{\epsilon^{(n)}_\alpha} \frac{\overrightarrow{\partial^l}}{\partial\lambda^\alpha_{(n)}} \frac{\overrightarrow{\partial^l}}{\partial\lambda^*_{(n)\alpha}} , \tag{6.3}$$

a $n'th$-level Planck constant $\hbar_{(n)}$, and a $n'th$-level Planck operator

$$\mathsf{Pl}_{(n)} = -\left(\lambda^\alpha_{(n)}\lambda^*_{(n)\alpha}, \ \cdot \ \right)_{[n]} + \hbar_{(n)}\frac{\partial}{\partial\hbar_{(n)}} . \tag{6.4}$$

At even levels n, one has the following $n'th$-level generalization of Principle 3.1:

Principle 6.1 *The $W_{[n]}$-action should satisfy three principles:*

1. *Planck number conservation:* $\quad \forall i = 1, \ldots, n : \ \mathsf{Pl}_{(i)}(\frac{W_{[n]}}{\hbar_{(n)}}) = 0.$

2. *The Quantum Master Equation:* $\Delta_{[n]} \exp\left[\frac{i}{\hbar_{(n)}}W_{[n]}\right] = 0.$

3. *The Hessian of $W_{[n]}$ has rank equal to half the number of fields $\Gamma^A_{[n]}$, i.e. $(n+1)N$ in the irreducible case.*

The $n'th$-level partition function is defined as

$$\mathcal{Z}^\Psi_{[n]} = \int [d\Gamma_{[n-1]}][d\lambda_{(n)}] \, \rho_{[n-1]} \, e^{\frac{i}{\hbar}(W^\Psi_{[n]}+X_{[n-1]})}\Big|_{\lambda^*_{(n)}=0} , \tag{6.5}$$

with

$$e^{\frac{i}{\hbar}W^\Psi_{[n]}} = e^{-[\overrightarrow{\Delta}_{[n]},\Psi]}e^{\frac{i}{\hbar}W_{[n]}} . \tag{6.6}$$

It follows from standard arguments that the partition function $\mathcal{Z}^\Psi_{[1]}$ does not depend on the gauge fermion Ψ. At odd levels n, there is a similar story where the rôles of W and X are exchanged, up to a relative sign,

$$e^{\frac{i}{\hbar}X^\Psi_{[n]}} = e^{[\overrightarrow{\Delta}_{[n]},\Psi]}e^{\frac{i}{\hbar}X_{[n]}} . \tag{6.7}$$

Again the Planck number conservation limits the number of possible $n'th$-level structure functions in the action

$$W_{[n]} = G_{(n-1)\alpha}\lambda^\alpha_{(n)} + (i\hbar_{(n)})H_{[n-1]} + \mathcal{O}(\lambda^*_{(n)}) . \tag{6.8}$$

When one decomposes the $n'th$-level Quantum Master Equation in terms of the above $(n-1)'th$-level structure functions one generates a tower of equations; the first few equations read:

$$(G_{(n-1)\alpha}, G_{(n-1)\beta})_{[n-1]} = G_{(n-1)\gamma}U^\gamma_{(n-1)\alpha\beta} , \tag{6.9}$$

$$(\Delta_{[n-1]}G_{(n-1)\beta}) - (H_{[n-1]}, G_{(n-1)\beta})_{[n-1]} = (-1)^{\epsilon^{(n)}_\alpha}U^\alpha_{(n-1)\alpha\beta} + G_{(n-1)\alpha}V^\alpha_{(n-1)\beta} , \tag{6.10}$$

$$-(\Delta_{[n-1]}H_{[n-1]}) + \frac{1}{2}(H_{[n-1]}, H_{[n-1]})_{[n-1]} = V^\alpha_{(n-1)\alpha} - G_{(n-1)\alpha}\tilde{G}^\alpha_{(n-1)} . \tag{6.11}$$

46

There is in general no covariant on-shell expression for the higher-level $H_{[n-1]}$ with $n \geq 2$, although partial results exist [5].

6.2 Recursive Reduction

The reduction from $n'th$ to $(n-1)'th$-level can be demonstrated as follows:

1. Go to coordinates

$$\Gamma^A_{[n]} \quad \to \quad \left\{ \Gamma^A_{[n-2]}; \lambda^\alpha_{(n-1)}, \lambda^*_{(n-1)\alpha}; \lambda^\alpha_{(n)}, \lambda^*_{(n)\alpha} \right\} \tag{6.12}$$

with Darboux coordinates at the last two levels n and $n-1$, and with a measure density $\rho_{[n-1]} \to \rho_{[n-2]}$.

2. Choose the gauge-fixing functions $G_{(n-1)\alpha} = \frac{\hbar_{(n-1)}}{\hbar} \lambda^*_{(n-1)\alpha}$, so that $U^\gamma_{(n-1)\alpha\beta} = 0$. In this case the eq. (6.10) yields that

$$H_{[n-1]} \quad = \quad K_{[n-1]} - \lambda^*_{(n-1)\alpha} \bar{V}^\alpha_{(n-1)\beta} \lambda^\beta_{(n-1)} , \tag{6.13}$$

where

$$\bar{V}^\alpha_{(n-1)\beta} \quad \equiv \quad \int_0^1 dt \, V^\alpha_{(n-1)\beta}(\Gamma_{[n-2]}; t\lambda_{(n-1)}, \lambda^*_{(n-1)}; \hbar_{[n-1]}) , \tag{6.14}$$

and where the integration "constant" $K_{[n-1]} = K_{[n-1]}(\Gamma_{[n-2]}; \lambda^*_{(n-1)}; \hbar_{[n-1]})$ is independent of $\lambda^\beta_{(n-1)}$.

3. Define the $(n-2)'th$-level action as

$$W_{[n-2]} \quad = \quad (i\hbar_{(n-2)}) \, H_{[n-1]} \Big|_{\lambda^*_{(n-1)}=0} , \tag{6.15}$$

if n is even. (If n is odd, exchange $X \leftrightarrow W$.) The action $W_{[n-2]}$ defined this way does not depend on $\hbar_{(n-1)}$, because of Planck number conservation $\mathsf{Pl}_{(n-1)}(W_{[n-2]}) = 0$, and moreover $W_{[n-2]}$ satisfies the $(n-2)'th$-level Quantum Master Equation

$$\Delta_{[n-2]} \exp\left[\frac{i}{\hbar_{(n-2)}} W_{[n-2]} \right] \quad = \quad 0 , \tag{6.16}$$

as a result of (one of the consequences of) the $n'th$-level Quantum Master Equation eq. (6.11). Note that other consequences of the $n'th$-level Quantum Master Equation would in general impose other conditions on $W_{[n-2]}$. (If the $(n-2)'th$-level action $W_{[n-2]}$ is already known independently, the eq. (6.15) should be interpreted as a boundary condition.)

4. Choose the gauge fermion Ψ independent of the $n'th$-level Lagrange multiplier antifields $\lambda^*_{(n)\alpha}$. Then Ψ is independent of the Lagrange multipliers $\lambda^\alpha_{(n)}$ as well, because of Planck number conservation $\mathsf{Pl}_{(n)}(\Psi) = 0$. (Here we have implicitly assumed that there are only non-negative powers of $n'th$-level objects $\{\Gamma^A_{(n)}; \hbar_{(n)}\}$ present inside Ψ.) Using the symmetry (1.1) of the $\Delta_{[n]}$-operator, one derives

$$\begin{aligned}
\mathcal{Z}^\Psi_{[n]} &= \int [d\Gamma_{[n]}] \, \rho_{[n-2]} \, e^{\frac{i}{\hbar} W_{[n]}} \, e^{[\vec{\Delta}_{[n]}, \Psi]} \delta(\lambda^*_{(n)}) e^{\frac{i}{\hbar} X_{[n-1]}} \\
&= \int [d\Gamma_{[n]}] \, \rho_{[n-2]} \, e^{\frac{i}{\hbar}(\lambda^*_{(n-1)\alpha} \lambda^\alpha_{(n)} + W_{[n-2]})} \, \delta(\lambda^*_{(n)}) \, e^{[\vec{\Delta}_{[n]}, \Psi]} e^{\frac{i}{\hbar} X_{[n-1]}} \\
&= \int [d\Gamma_{[n-2]}][d\lambda_{(n-1)}] \, \rho_{[n-2]} \, e^{\frac{i}{\hbar}(W_{[n-2]} + X^\Psi_{[n-1]})} \Big|_{\lambda^*_{(n-1)}=0} \quad = \quad \mathcal{Z}^\Psi_{[n-1]} , \tag{6.17}
\end{aligned}$$

which completes the reduction step.

The $n'th$-level formalism can also be set up in the case of reducible gauge-fixing constraints and second-class constraints. In the reducible case one introduces the relevant stages of minimal and non-minimal fields, in accordance with the general field-antifield prescription, as we saw in Section 5.

Hence the multi-level formalism consists in recursively building master actions $W_{[0]} \equiv W$, $X_{[1]} \equiv X$, $W_{[2]}$, $X_{[3]}$, ... By zig-zagging through the W- and X-parts it becomes a simple matter to re-use the constructions of the previous levels and to create a manifestly gauge-independent formalism.

7 Conclusion

Driven by the wish to develop the Lagrangian quantization program into its most general and axiomatized formulation, we have in this paper focused on three aspects. First, we have given a more fully geometric description of the multi-level formalism, in all generality. Second, we have explored the new symmetric formulation which puts the "action" W on the same footing as the "gauge-fixing" X to yield the full quantum action that enters in the Boltzmann factor of the functional integral. One particular aspect of this symmetry concerns the algebras behind the Master Actions W and X. On one hand there is the gauge-generating algebra behind W, which is associated with gauge symmetries of the classical action, and is known to accommodate both open and reducible gauge-algebras. On the other hand there is the gauge-fixing algebra behind X, which is a quantum mechanical feature, with no analog at the level of the classical theory, that usually carries just an irreducible algebra. We have in detail demonstrated how to permit a reducible gauge-fixing algebra, and calculated the associated measure or gauge volume from first principles, namely by solving the Quantum Master Equation. Several consistency checks on the formalism were performed along the way by reduction methods. Third, we included an extensive discussion of antisymplectic second-class constraints in the field-antifield formalism, and demonstrated manifest invariance under reparametrization of the second-class constraints. Second-class constraints hold surprisingly many features in common with the gauge-fixing constraints G_α, and often do they appear side-by-side in the formulas. In this way, the second-class constraints merge effortlessly with the multi-level formalism, even in the case of reducible gauge-fixing algebras.

While there are still many more aspects of the new and more axiomatic formulation of the Lagrangian quantization prescription that need to be explored, it is already now apparent that there is a rich and beautiful algebraic structure behind. This algebra has its root in the one single object from which all is derived: the Grassmann-odd nilpotent Δ-operator, and its associated Quantum Master Equation.

NOTE ADDED: After the paper appeared on the archive we became aware of Ref. [17] where a reducible gauge-fixing X arose in a superfield context.

CORRECTION ADDED AFTER PUBLICATION IN NUCLEAR PHYSICS B: From three lines above eq. (4.33) down to three lines below eq. (4.34): The three consecutive sentences, which start with "For every system of unitarizing coordinates ...", should be discarded. This is because the statement that eq. (4.34) can be solved recursively to all orders in Θ, is, in general, incorrect. Nevertheless, transversal coordinate systems do exist locally, which is all that is needed for the ensuing discussion.

References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B** (1981) 27.

[2] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D28** (1983) 2567 [E: **D30** (1984) 508].

[3] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234** (1984) 106.

[4] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. **A8** (1993) 2333.

[5] I.A. Batalin and I.V. Tyutin, Mod. Phys. Lett. **A8** (1993) 3673; *ibid.* **A9** (1994) 1707; Amer. Math. Soc. Transl. **2.177** (1996) 23.

[6] I.A. Batalin, R. Marnelius and A.M. Semikhatov, Nucl. Phys. **B446** (1995) 249.

[7] I.A. Batalin, K. Bering and P.H. Damgaard, Phys. Lett. **B389** (1996) 673.

[8] B.L. Voronov and I.V. Tyutin, Theor. Math. Phys. **50** (1982) 218; *ibid.* **52** (1982) 628.

[9] O.M. Khudaverdian and Th. Voronov, Lett. Math. Phys. **62** (2002) 127; O.M. Khudaverdian, Contemp. Math. **315** (2002) 199; O.M. Khudaverdian, Commun. Math. Phys. **247** (2004) 353.

[10] A. Schwarz, Commun. Math. Phys. **155** (1993) 249.

[11] O.M. Khudaverdian and A.P. Nersessian, Mod. Phys. Lett. **A8** (1993) 2377.

[12] K. Bering, Ph.D. Thesis, Uppsala University, Sweden, 1997; I.A. Batalin and I.V. Tyutin, Theor. Math. Phys. 114 (1998) 198.

[13] O.M. Khudaverdian, prepared for the *International Workshop On Geometry And Integrable Models 4-8 October 1994, Dubna, Russia.* In *Geometry and Integrable models*, (P.N. Pyatov and S. Solodukhin, Eds.), World Scientific, 1996.

[14] I.A. Batalin, K. Bering and P.H. Damgaard, Phys. Lett. **B408** (1997) 235.

[15] I.A. Batalin and R. Marnelius, Nucl. Phys. **B511** (1998) 495.

[16] A. Schwarz, Lett. Math. Phys. **2** (1978) 247; Commun. Math. Phys. **67** (1979) 1.

[17] A.A. Reshetnyak, Russ. Phys. J. **47** (2004) 1026.